Journal
of
The Franklin Institute
Devoted to Science and the Mechanic Arts

Vol. 255 JUNE, 1953 No. 6

RADIO TECHNOLOGY AND THE THEORY OF NUMBERS *

BY

BALTH. VAN DER POL [1]

The subject of my lecture combines two sciences which, at first sight, may perhaps seem utterly unrelated. In fact, I do not know of any literature which suggests a relationship between these two domains. I refer on the one hand to *Radio Technology* in its widest sense, a subject about which we have all heard a great deal and with which we are personally acquainted, if only through our use of radio sets at home; and on the other hand, to the *Theory of Numbers*, a part of pure mathematics, and perhaps even the purest part of it. Gauss, the great German mathematician, is said to have stated that "if mathematics is the Queen of Sciences, then Number Theory is the Queen of Mathematics." Number Theory is concerned for a large part with the natural numbers 1, 2, 3, ···, etc. These natural numbers have an unexpected wealth of properties, and some of the theorems discovered about them belong to the deepest regions human intellect has been able to penetrate. This led another German mathematician, Kronecker, to say "God created the integers, all the rest is the work of men."

We are, of course, to a certain extent, all acquainted with the natural numbers, for example, when in the morning we pay the milkman in Krone and Øre. But the natural numbers extend beyond a hundred and beyond a million, and even beyond the greatest numbers we encounter in astronomy. Hence the material at the disposal of those who study the Theory of Numbers extends towards infinity. And each of these natural numbers is either prime or composite. The primes are not divisible by any other integer apart from unity and the prime itself. Primes are, for instance, 2 or 3 or 5 or 7 or ··· 170,141,183,460,469,231,

* Lecture delivered before the Danish Technical Academy, January 20, 1953, on the occasion of the award to the author of the Valdemar Poulsen Gold Medal.

[1] Director, International Radio Consultative Committee (C.C.I.R.), Geneva, Switzerland.

(Note—The Franklin Institute is not responsible for the statements and opinions advanced by contributors in the JOURNAL.)

731,687,303,715,884,105,727, which latter number contains 39 digits when written in the usual decimal system.

It is, of course, not easy to verify that such a large number is not composite, that is, that it is not divisible by any smaller number except unity. And yet the American mathematician, Dr. D. H. Lehmer, told me recently at Los Angeles, when I visited his mathematical laboratory there, that he had discovered an even greater prime number. It certainly is the greatest number of which we are sure that it is indeed a prime. It is:

$$2^{1279} - 1$$

This number, written out in the usual decimal system, contains 397 digits. It took Dr. Lehmer's modern electrical digital computer 14 minutes to determine this number, whereas it would have taken a human being some 125 years to do so.

Now, what relation exists between Radio Technology and the Theory of Numbers?

Before I can answer this question, I would like to consider, in an even more general sense, the relation between mathematics on the one hand and physics and technology on the other. History shows us that mathematics has, to a great extent, been developed by pure mathematicians who were under the spell of the beauty of it; who were entranced by its mysterious generality and who spent their life discovering new facts and relations in this very wide and wonderful domain. Whether their results could be applied to astronomy, physics, chemistry, or say, technology, as a rule did not interest them primarily. Their main concern was to make their theorems cover cases of a more and more abstract nature, and generality in their work was their main purpose. In this sense we have to understand the famous saying of the English mathematician and philosopher, Sir Bertrand Russell, to the effect that "a mathematician is never so happy as when he does not know what he is talking about."

Some mathematicians even went as far as to be proud of the fact that none of their results could be applied in practice. It is said that one great mathematician remarked that "Bessel functions are beautiful functions in spite of their many applications."

Now, physics and technology were developed to a certain extent independently of pure mathematics. It is therefore natural that at a given moment the physicists or technologists discovered, to their surprise, that precisely the mathematical tools and methods they needed to solve their problems had already been fully developed by pure mathematicians who, at the time, did not have the slightest notion of, or even interest in, a possible practicable application of their results.

The main purpose of my lecture is to show that, in my view, both physics and radio technology have reached a stage where methods and

results properly belonging to the Theory of Numbers can be applied, and are beginning to be applied with much success, to the solution of problems typical of physics or radio technology.

I know that some mathematicians do not agree with this view. I hesitate to quote in this respect the great English mathematician, G. H. Hardy, and this for two reasons: first, because I have a great debt towards him for what he has given us in his beautiful mathematical researches, especially in the Theory of Numbers, and second, because he cannot contradict me any more. If I therefore venture to quote him, I do so with all due respect. In the otherwise charming and sometimes even pathetic essay (1) [2] which he wrote not long before his death, he says (pages 60–61):

> . . . If the Theory of Numbers could be employed for any practical and obviously honourable purpose, if it could be turned directly to the furtherance of human happiness or the relief of human suffering, as physiology and even chemistry can, then surely neither Gauss nor any other mathematician would have been so foolish as to decry or regret such applications. But science works for evil as well as for good (and particularly, of course, in time of war); and both Gauss and lesser mathematicians may be justified in rejoicing that there is one science at any rate, and that their own, whose very remoteness from ordinary human activities should keep it gentle and clean.

Again, another great mathematician and number theorist, Edmund Landau, wrote on page 21 of the first volume of his standard work (2): "Gordan pflegte etwa zu sagen: 'Die Zahlentheorie ist nützlich, weil man nämlich mit ihr promovieren kann'." [3]

These views may have been right and more or less complete at the time they were written, but it is my opinion that today physics and radio technology have reached a stage where ideas, methods and results, typically belonging to advanced Number Theory, can be applied with great success. Let me give you a few examples illustrating my thesis:

(*a*) In X-ray spectroscopy of cubic crystals it has been noted that certain reflections are absent. One finds reflections corresponding to the following integers: 1, 2, 3, 4, 5, 6, —, 8, 9, 10, 11, 12, 13, 14, —, 16, 17, $\cdots$ and it appeared that the missing numbers n are of the form $n = 4^k(8m + 7)$, where k and m are positive integers. Now Number Theory tells us that these are numbers which cannot be represented as the sum of three or less squares. There seems to be little doubt that the fact quoted is closely related to the three dimensions of our Euclidean space. I have inserted in Table I a short list of the natural numbers up to 28, with their representations as a sum of four or less squares. More than four squares are never required, but this number four is abso-

[2] The bold face numbers in parentheses refer to the references appended to this paper.

[3] "Gordan used to say: 'Number Theory is useful because it is a subject in which one may obtain a doctor's degree.'"

lutely necessary in the case of 7, 15, 23, $\cdots$, which are all of the form quoted above.

TABLE I.—*The Representation of the Natural Numbers by Four or Less Squares.*

$1 = 1^2$
$2 = 1^2 + 1^2$
$3 = 1^2 + 1^2 + 1^2$
$4 = 2^2 = 1^2 + 1^2 + 1^2 + 1^2$
$5 = 2^2 + 1^2$
$6 = 2^2 + 1^2 + 1^2$
$7 = 2^2 + 1^2 + 1^2 + 1^2$
$8 = 2^2 + 2^2$
$9 = 3^2 = 2^2 + 2^2 + 1^2$
$10 = 3^2 + 1^2 = 2^2 + 2^2 + 1^2 + 1^2$
$11 = 3^2 + 1^2 + 1^2$
$12 = 2^2 + 2^2 + 2^2 = 3^2 + 1^2 + 1^2 + 1^2$
$13 = 3^2 + 2^2 = 2^2 + 2^2 + 2^2 + 1^2$
$14 = 3^2 + 2^2 + 1^2$
$15 = 3^2 + 2^2 + 1^2 + 1^2$
$16 = 4^2$
$17 = 4^2 + 1^2 = 3^2 + 2^2 + 2^2$
$18 = 3^2 + 3^2 = 4^2 + 1^2 + 1^2 = 3^2 + 2^2 + 2^2 + 1^2$
$19 = 3^2 + 3^2 + 1^2 = 4^2 + 1^2 + 1^2 + 1^2$
$20 = 4^2 + 2^2 = 3^2 + 3^2 + 1^2 + 1^2$
$21 = 4^2 + 2^2 + 1^2$
$22 = 3^2 + 3^2 + 2^2 = 4^2 + 2^2 + 1^2 + 1^2$
$23 = 3^2 + 3^2 + 2^2 + 1^2$
$24 = 4^2 + 2^2 + 2^2$
$25 = 5^2 = 4^2 + 3^2 = 4^2 + 2^2 + 2^2 + 1^2$
$26 = 5^2 + 1^2 = 4^2 + 3^2 + 1^2 = 3^2 + 3^2 + 2^2 + 2^2$
$27 = 3^2 + 3^2 + 3^2 = 5^2 + 1^2 + 1^2 = 4^2 + 3^2 + 1^2 + 1^2$
$28 = 4^2 + 2^2 + 2^2 + 2^2 = 3^2 + 3^2 + 3^2 + 1^2 = 5^2 + 1^2 + 1^2 + 1^2$

(*b*) The theory of potentials inside a crystal lattice (for example, the well known Madelung constant) can best be derived with the help of the elliptic theta-functions which at the same time form the basis of the analytical derivation in Number Theory of the representation of an integer as the sum of squares. In fact the two derivations are completely analogous.

(*c*) Several aspects of Planck's radiation formula and of the Einstein-Bose and Fermi statistics are closely related to Riemann's zeta-function $\zeta(s)$ which is defined by:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx \qquad (\mathrm{Re}\, s > 1) \quad (1)$$

and which is at the base of practically all arithmetical investigations on the distribution of the prime numbers. This fact seems to have been realized in the United States to the extent that, during the war, even purely mathematical researches on the properties of the ζ-function were

classified as security subjects (verbal communication from Dr. Norbert Wiener).

(*d*) Another number-theoretic function, which is now beginning to play an important part in physics, is the partition function $p(n)$. It is defined by the generating function:

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = 1 + \sum_{1}^{\infty} p(n)x^n. \qquad (|x| < 1) \qquad (2)$$

The symbol $p(n)$ enumerates the number of ways in which the integer n can be written as the sum of any positive integers, repetitions being allowed. Thus, for example:

$$\begin{aligned} 6 &= 5+1 = 4+2 = 4+1+1 = 3+3 = 3+2+1 \\ &= 3+1+1+1 = 2+2+2 = 2+2+1+1 \\ &= 2+1+1+1+1 = 1+1+1+1+1+1 \end{aligned}$$

which shows eleven different ways of writing 6 as a sum of positive integers, and therefore $p(6) = 11$.

Applications of this partition function $p(n)$ and its asymptotic behavior are now being made in different parts of physics, among other things in the theory of the growth of crystals (3) and are used to explain the irregularities which sometimes occur on crystal surfaces.

These are a few examples borrowed from theoretical physics where modern problems can be successfully attacked with methods and results belonging to the field of Number Theory.

But also in radio fruitful applications of this most abstract part of mathematics can be made. Let me give you again a few instances.

(*e*) It is well known that when a periodic emf. is acting on a system which is generating *relaxation oscillations*, the system will, under certain circumstances, be forced to oscillate with a frequency which is a submultiple of the forcing frequency. Extensive use is made of these oscillations in television. Thus, we are confronted with a frequency demultiplication or frequency *division*. Well now, divisibility of numbers is a characteristic part of Number Theory, and perhaps it is even one of its most important chapters. It appears not improbable that here again arithmetical methods could help in further elucidating this highly complex phenomenon which is used extensively in practice. At the same time, several mathematicians of different nationalities are still pursuing theoretical research on this subject, among them Miss Mary L. Cartwright in Cambridge (England) (4).

(*f*) Let me give you next another example which has even a very classical flavor. It concerns the problem of the capacitance between two conducting spheres. A very elegant solution of this problem was given 100 years ago by Lord Kelvin with the aid of his theory of electrical images (5).

However, it is perhaps not generally known that Kelvin's expression for the capacitance $C_{a,b}$ of two mutually external spheres of radii a and b, respectively, and with their centers at a distance c apart, can be written in the form:

$$C_{a,b} = \frac{EI}{c} \sum_{n=1}^{\infty} \left\{ d(n) - d\left(\frac{n}{2}\right) \right\} \alpha^n, \tag{3}$$

where

$$\alpha = \frac{E - I}{E + I}$$

and E and I are the length of the external and internal tangents, respectively, to the circles obtained by cutting the sphere with a plane through their centers. In (3) we find the function $d(n)$, characteristic of Number Theory, representing the number of divisors of n. To take an example, $d(6) = 4$, because the number 6 has four divisors, *viz.* 1, 2, 3 and 6. Incidentally, in (3), $d\left(\frac{n}{2}\right)$ has to be replaced by zero for any odd n. Hence, in writing down some of the terms in (3) explicitly, we have:

$$C_{a,b} = \frac{EI}{c} \{\alpha + \alpha^2 + 2\alpha^3 + \alpha^4 + 2\alpha^5 + 2\alpha^6 + 2\alpha^7 + \alpha^8 + 3\alpha^9 + 2\alpha^{10} + 2\alpha^{11} + 4\alpha^{12} + \cdots + 2\alpha^{6089} + 16\alpha^{6090} + 2\alpha^{6091} + \cdots\}. \tag{3a}$$

When it was found that $C_{a,b}$ could be written in the form (3) or (3*a*), with the coefficients $d(n)$ peculiar to Number Theory, the analogy of this problem with the famous Dirichlet problem was recognized. The latter consists of the calculation of the behavior of the sum $\sum_1^N d(n)$ when N tends towards infinity (6).

As far as we know the present literature does not contain any formulae giving an approximation for this capacitance $C_{a,b}$ when the spheres are very close together, that is, at a distance d which is small compared with both radii. In this case the variable α is close to unity and our series (3) or (3*a*) then converges very slowly indeed. The expression (3), however, enabled us to apply mathematical methods which are peculiar to Number Theory. Thus in the case of two spheres of equal radii a and a small distance d apart the following simple expression was found for their mutual capacitance:

$$C_{a,a} \approx \tfrac{1}{2} a \log\left(\frac{a}{d}\right). \tag{4}$$

A complete derivation of (4) is given in the Appendix.

Thus we see how, even in a very classical electrical problem, analytical methods borrowed from modern Number Theory may be used

(*g*) But there is another typical—and very fundamental—radio problem which I have never seen plainly explained in the literature and which also leads us to borrow results from Number Theory.

Suppose a voltage v consisting of the superposition of two cosine functions of time, and which is therefore of the form

$$v = a \cos \omega_1 t + b \cos \omega_2 t, \tag{5}$$

is applied to a detector with a square law characteristic representable by

$$i = 2\alpha v^2. \tag{6}$$

Substitution of (5) into (6) leads to:

$$i = \alpha\{a^2(1 + \cos 2\omega_1 t) + b^2(1 + \cos 2\omega_2 t) + 2ab\{\cos(\omega_1 + \omega_2)t + \cos(\omega_1 - \omega_2)t)\}.$$

This formula clearly shows that the following frequencies are present in the current:

$$0, \quad 2\omega_1, \quad 2\omega_2, \quad \omega_1 + \omega_2, \quad \omega_1 - \omega_2.$$

When, however, the detector characteristic is such that it is only representable by a more complicated transcendental function than a parabola, such as is the case with the so-called "linear" detection, or when $i = e^{\beta v}$, then the frequencies—including the negative ones—present in the resulting current, generally comprise all possible combination tones, *viz.*:

$$m\omega_1 + n\omega_2. \tag{7}$$

Here both m and n run independently through all positive and negative integers, including zero. When further ω_1 and ω_2 are commensurate (which is always at least approximately true), and using a proper time scale, (7) can be replaced by:

$$ma + nb,$$

where now a and b are integers. A combination of integers such as expressed by (7*a*) is written $[a, b]$ and is called in Number Theory a "modulus" (**7**). Incidentally, in connection with this, Fueter quotes a book by Wilhelm Fliess, entitled "Zum Ablauf des Lebens," in which the latter maintains that the modulus [23, 28] represents all the numbers important in human life. Fliess shows, for example, that:

$$\begin{aligned} 1 \times 28 - 1 \times 23 &= 5 \\ 3 \times 23 - 2 \times 28 &= 13 \\ 5 \times 23 - 4 \times 28 &= 3 \\ 5 \times 28 - 6 \times 23 &= 2 \end{aligned}$$

the resulting numbers 5, 13, 3, 2 all representing ages of special importance in human life. However, as Fueter clearly points out, the numbers represented by the modulus [23, 28] or the numbers

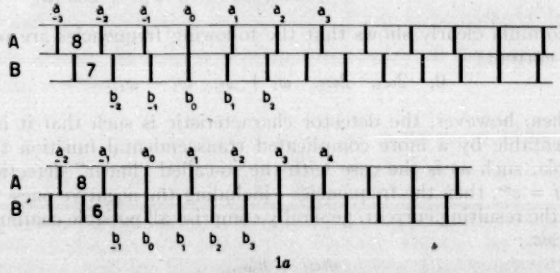
$$23n + 28m,$$

where m and n run through all integers positive and negative, in point of fact cover every whole number from minus infinity to plus infinity. All we have to do, to represent any given integer, is to choose a proper combination of m and n.

This is the case for the base 23 and 28 because the highest common divisor of 23 and 28 is unity. If we had taken, for example, the modulus

$$16n+20m$$

the numbers generated by this formula are obviously all multiples of 4 because the highest common divisor of 16 and 20 is 4.



1a

FIG. 1. Vernier showing the distribution of the combination tones when two sinusoidal oscillations are applied to a detector.

Returning now to our radio problems of the combination tones, as given by (7a), the general theory tells us that the modulus (7a) represents *all positive and negative multiples of the highest common divisor of the numbers* a *and* b, or (with a proper transformation of the time scale) of ω_1 and ω_2. Therefore the combination tones obtained with a transcendental detector are all nicely and equally spaced at a frequency distance corresponding to the highest common divisor of a and b, provided ω_1 and ω_2 are commensurate, otherwise the combination tones are "dense everywhere." We can thus, for example, state beforehand how many combination tones there will be found in a given frequency band, this number depending only on the width of the band and not on its position in the frequency spectrum.

The theorem mentioned, *viz.* that the numbers generated by

$$ma + nb, \qquad (7a)$$

where a and b are fixed integers, but m and n run, independently, through all positive and negative integers, are nothing else than the multiples of the highest common divisor of two numbers a and b, is of very fundamental importance in Number Theory. In fact, the highest common divisor of two numbers a and b can best be defined as the minimum positive value of the expression (7a) obtainable by a proper selection of m and n.

It is of interest to note here that this fundamental theorem is related to what is known in technology as the *vernier* or *nonius*. It is best explained with the aid of two linear scales. In Fig. 1 we have two scales A and B. The distance between successive markings (a_n, a_{n+1}) on scale A is 8 units, and that between successive markings (b_m, b_{m+1}) on scale B is 7 units. Now, we first place the scales in such a way that the—arbitrarily chosen—zero marks (a_0 and b_0) coincide. Then, it will be clear that the distance between any mark a_n on scale A and any other mark b_m on scale B is

$$8n - 7m \text{ units.}$$

In particular, we see clearly from the figure that

$$\begin{aligned} a_1 - b_1 &= 1 \text{ unit} \\ a_2 - b_2 &= 2 \text{ units} \\ a_3 - b_3 &= 3 \text{ units} \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

and hence all integers can thus be generated. The smallest distance, except zero (here 1) is the highest common divisor of 8 and 7.

Consider next Fig. 1a where we have again scale A, while scale B is replaced by scale B', in which the markings are six units apart.

The smallest distance (apart from zero) of a marking on scale A and of one on scale B' is now 2, for example, between a_1 and b_1, or between b_3 and a_2. All the other distances are multiples of 2, which is itself the highest common divisor of 8 and 6.

We now come to our last example taken from radio technology.

(h) It is the so-called "sawtooth" function, which is depicted in Fig. 2. We denote it by Sa(x). It reaches its highest values $\frac{1}{2}$ (and at

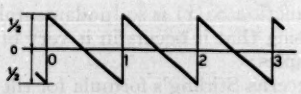


FIG. 2. The sawtooth function $y = \text{Sa}(x)$.

the same time its lowest values $-\frac{1}{2}$) at $x = \cdots -3, -2, -1, 0, 1, 2, 3, \cdots$ and therefore shows a jump of magnitude unity at the integer points, whereupon every time it falls linearly.

This function of time is used in any television transmitter and receiver in order to scan periodically the lines and frames of the picture. Important technical developments have recently been achieved to generate sawtooth functions of proper shape and behavior experimentally. But, at the same time, this very sawtooth function $\mathrm{Sa}(x)$, which we use so extensively in radio technology, is the most fundamental function in Number Theory. This may be clarified with the

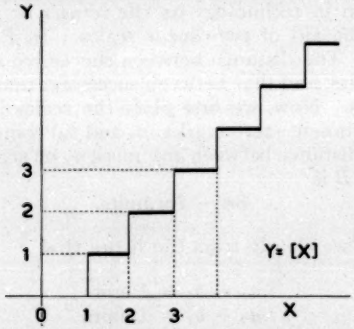


FIG. 3. Graph of the function $y = [x]$ representing the integers.

aid of a very simple construction. When we represent the integers graphically we obtain Fig. 3. This staircase function is denoted in Number Theory by

$$y = [x]. \tag{8}$$

If we subtract from (8) the simple function $y = x$ and add $\frac{1}{2}$, we obtain at once the sawtooth function $\mathrm{Sa}(x)$

$$\mathrm{Sa}(x) = [x] - x + \tfrac{1}{2}, \tag{9}$$

which is periodic with a period unity, and therefore has the property

$$\mathrm{Sa}(x + 1) = \mathrm{Sa}(x).$$

This sawtooth function $\mathrm{Sa}(x)$ is so fundamental in Number Theory that it is not surprising that it occurs in it very often indeed. May I give you a few examples:

The first one concerns Stirling's formula for the factorial

$$\Pi(x) = \Gamma(x + 1).$$

We have the well known expression:

$$\Pi(x) = \sqrt{2\pi x}\, x^x e^{-x} e^{\mu(x)} \qquad (x > 0), \tag{10}$$

where $\mu(x)$ tends towards zero for $x \to \infty$. This latter, rather complicated, function $\mu(x)$ has a very elegant and simple representation in terms of the sawtooth function; it is:

$$\mu(x) = \int_0^\infty \frac{\mathrm{Sa}(u)}{u + x}\, du. \tag{11}$$

But (11) can be generalized further if we introduce Hurwitz's generalization (8) $\zeta(s, x)$ of the Riemann zeta-function $\zeta(s)$. It is defined, for $\mathrm{Re}\, s > 1$, by

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s} \qquad \left.\begin{matrix} \mathrm{Re}\, s > 1 \\ x > 0 \end{matrix}\right\}$$

so that

$$\zeta(s, 1) = \zeta(s).$$

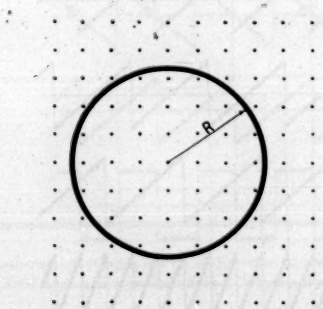


FIG. 4. The number of lattice points within a circle of radius R.

The function $\zeta(s, x)$ can be analytically continued for $\mathrm{Re}\, s < 1$, and it can again be represented with the aid of the sawtooth function $\mathrm{Sa}(x)$ as follows:

$$\frac{1}{s}\left\{\zeta(s, x + 1) - \frac{1}{(s - 1)x^{s-1}} + \frac{1}{2x^s}\right\} = \int_0^\infty \frac{\mathrm{Sa}\, u}{(u + x)^{s+1}}\, du \qquad \left.\begin{matrix} \mathrm{Re}\, s > -1 \\ x > 0 \end{matrix}\right\}. \tag{12}$$

Thus (12) is a generalization of (11) to which it is reduced if in (12) we take $s = 0$.

Again, if in (12) we restrict s to the range $-1 < \mathrm{Re}\, s < 0$, x may be taken zero, and we thus obtain for the Riemann's zeta function the

elegant expression:

$$\frac{1}{s}\zeta(s) = \int_0^\infty \frac{\mathrm{Sa}(u)}{u^{s+1}}\,du, \qquad -1 < \mathrm{Re}\, s < 0 \tag{13}$$

a function to the study of which so much has been contributed by the great Danish mathematician Harold Bohr, and which is at the basis of the whole theory of the distribution of prime numbers. The expressions (11), (12) and (13) give only a few examples of the occurrence of the sawtooth function in analysis.

Another example, typical of Number Theory, is the following:

Consider a square lattice as depicted in Fig. 4, where the distance between different points is taken as unity. If we draw a circle, with its center at any lattice point and with a radius of length R, the number

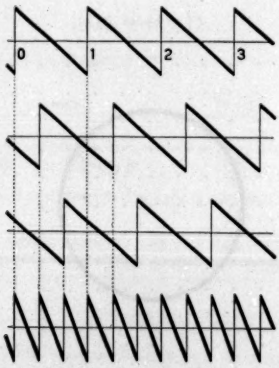


FIG. 5. The addition of three sawtooth functions shifted over $\frac{0}{3}$, $\frac{1}{3}$ and $\frac{2}{3}$ of a period, respectively, yield another sawtooth function having three times the original frequency.

of lattice points within the circle will obviously increase with R. A classical problem in Number Theory is to calculate the number $A(R^2)$ of lattice points within a circle of radius R. It will be clear at the outset that, if R becomes very large, this number will approximately equal the area of the circle, that is, πR^2. The correction to be applied to this approximation in order to obtain the exact number can be expressed elegantly as follows in series of sawtooth functions, where $R = \sqrt{x}$:

$$\tfrac{1}{4}\{A(x) - \pi x\} = \mathrm{Sa}(x) - \mathrm{Sa}\left(\frac{x}{3}\right) + \mathrm{Sa}\left(\frac{x}{5}\right) - \mathrm{Sa}\left(\frac{x}{7}\right) + \cdots.$$

The sawtooth function $\mathrm{Sa}(x)$ also has properties which are visible at once from a graphical construction. In fact, the periodic sawtooth

function $\mathrm{Sa}(x)$ is in some respects simpler than, say, the sine function $\sin 2\pi x$, although its Fourier representation

$$\mathrm{Sa}(x) = \sum_{k=-\infty}^{k=+\infty}{}' \frac{e^{2\pi ikx}}{2\pi ik} = \sum_{k=1}^{\infty} \frac{\sin 2\pi kx}{\pi k} \tag{14}$$

contains an infinity of sine terms.

One of the properties referred to is:

$$\sum_{k=1}^{n} \mathrm{Sa}\left(x - \frac{k}{n}\right) = \mathrm{Sa}(nx), \tag{15}$$

which is shown in Fig. 5, where $n = 3$. Expressed in words, this means that the superposition of n sawtooth functions, each one shifted over a phase difference $1/n$ with respect to the preceding one, yields a further

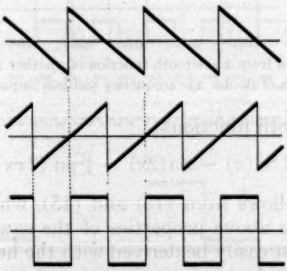


FIG. 6. The subtraction from a sawtooth function of another sawtooth function shifted over half a period gives a "square sine" function.

sawtooth function, but of n times the frequency. Properties of this sort surely have applications in radio practice. There are also other properties of the sawtooth function (see Fig. 6), which might be used experimentally, such as:

$$\mathrm{Sa}(x) - \mathrm{Sa}(x - \tfrac{1}{2}) = \tfrac{1}{2}\sqcap\!\!\sin(2\pi x), \tag{16}$$

where $\sqcap\!\!\sin(2\pi x)$ represents the "square sine" function which jumps from -1 to $+1$ at $x = \cdots, -1, 0, 1, 2, 3, \cdots$ and from $+1$ to -1 at $x = \cdots, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \cdots$ whereas it stays constant at 1 or -1 in the intervals between the jumps. It can also be written in the two forms

$$\sqcap\!\!\sin(2\pi x) = \frac{\sin 2\pi x}{|\sin 2\pi x|} = (-1)^{[2x]}.$$

It therefore has some analogy with the ordinary sine function, except that it behaves quite discontinuously.

It is not difficult, once again, to show (for example, graphically) that there exists also the following way (Fig. 7) to produce a $\sqcap\!\!\sqcup\text{in}\ (2\pi x)$

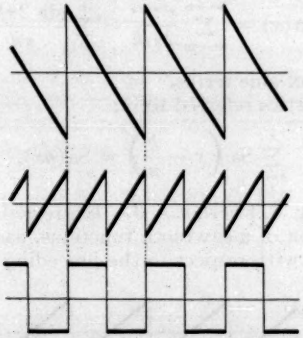


FIG. 7. The subtraction from a sawtooth function of another sawtooth function of half the amplitude but of double the frequency yields a "square sine" function.

function from sawtooth functions:

$$2\,\mathrm{Sa}(x) - \mathrm{Sa}(2x) = \tfrac{1}{2}\sqcap\!\!\sqcup\text{in}\ (2\pi x). \tag{17}$$

Incidentally, (17) follows from (16) and (15), when, in the latter, we take $n = 2$. All the above properties of the sawtooth functions and many others can most easily be derived with the help of the operational calculus (9).

It can also be shown that $\mathrm{Sa}(x)$ can be developed in a series of square sine functions of frequency 2^n. In fact, we have:

$$\mathrm{Sa}(x) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \cdot \sqcap\!\!\sqcup\text{in}\ (2^{n-1} \cdot 2\pi x) = \sum_{1}^{\infty} \frac{1}{2^{n+1}}\,(-1)^{[2^n x]}.$$

This synthesis of the sawtooth function $\mathrm{Sa}(x)$ from square sine functions is illustrated in Fig. 8. At the top we have five square sine functions a, b, c, d and e. The graph A is the synthesis of a and b, while B is the synthesis of a, b, c, d and e.

This development of the $\mathrm{Sa}(x)$ function has, contrary to its Fourier development (14), the interesting property that the so-called Gibbs' phenomenon is absent here. A similar synthesis of a sawtooth function from its ordinary Fourier components (sine functions) with the resulting Gibbs' phenomenon is shown in Fig. 9, where the "overshoots" at the discontinuities are clearly visible.

I also recently found the following functional relations of the sawtooth function:

$$\mathrm{Sa}(\mathrm{Sa}\,x) = -\,\mathrm{Sa}(x + \tfrac{1}{2})$$

and therefore:

$$\mathrm{Sa}\{\mathrm{Sa}(\mathrm{Sa}(x))\} = \mathrm{Sa}(x). \tag{18}$$

Thus (18) shows us that a sawtooth of a sawtooth of a sawtooth is again the original sawtooth function. Therefore any odd number of iterations always reproduces the original function.

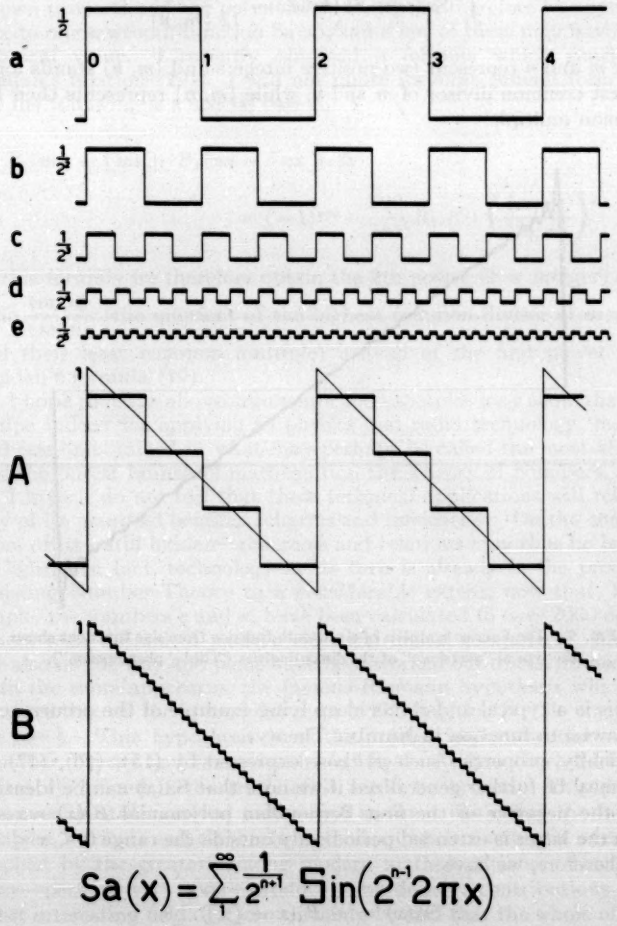


FIG. 8. The synthesis of a sawtooth function from "square sine" functions.

The relations expressed by (15), (16), (17) and (18) are all relatively simple, but there are much deeper relations involving the sawtooth function. For instance, Landau (Ref. 2, II, 170), proves the following interesting formula:

$$\int_0^1 \mathrm{Sa}(mx)\cdot\mathrm{Sa}(nx)dx = \frac{1}{12}\frac{(m,n)}{\{m,n\}}. \tag{19}$$

Here m and n represent two positive integers and (m, n) stands for the highest common divisor of m and n, while $\{m, n\}$ represents their least common multiple.

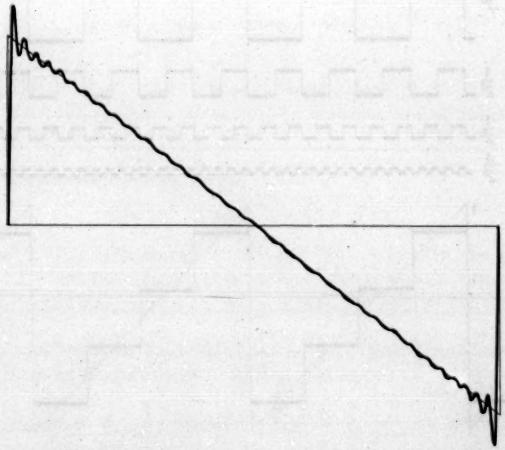


FIG. 9. The Fourier synthesis of a sawtooth function from sine functions shows the typical "overshoot" at the discontinuities ("Gibbs' phenomenon").

This is a typical and rather deep lying example of the occurrence of the sawtooth function in Number Theory.

Finally, properties such as those expressed by (15), (16), (17) and (19) may be further generalized if we note that $\mathrm{Sa}(x)$ can be identified with the negative of the first Bernoullian polynomial $B_1(x) = x - \frac{1}{2}$, when the latter is extended periodically outside the range $0 < x \leq 1$.

Therefore, we have:

$$\mathrm{Sa}(x) = -B_1(x - [x]). \tag{20}$$

In fact we can define the Bernoullian polynomials $B_n(x)$ with the aid of the generating function:

$$\frac{te^{tx}}{e^t - 1} = \sum_0^\infty \frac{B_n(x)}{n!} t^n.$$

Known properties of the polynomials $B_n(x)$ can therefore be applied at once to our sawtooth function $\mathrm{Sa}(x)$, and a few of these may have some technical value. I recently obtained a formula which generalizes Landau's relation (19). I will not quote its most general form here, but the following is a special case:

$$\int_0^1 B_k(mx - [mx]) \cdot B_k(nx - [nx]) \cdot dx = (-1)^{k+1} \cdot \frac{k!k!}{(2k)!} B_{2k}(O) \left(\frac{(m, n)}{\{m, n\}} \right)^k. \quad (21)$$

In this formula we therefore obtain the kth power (k = integer) of the ratio $\frac{(m, n)}{\{m, n\}}$ (the quotient of the highest common divisor of m and n, and their least common multiple) instead of the first power, as in Landau's formula (19).

I hope that the above arguments and examples may show that time is ripe indeed for applying to physics and radio technology, methods and results obtained in what may perhaps be called the most abstract and the purest branch of mathematics, the Theory of Numbers.

I myself do not feel that these technical applications will rob it of any of its manifold beauties, charms and intricacies. On the contrary, more of its—still hidden—theorems and relations may thus be brought to light. In fact, technology in its turn is already in the process of assisting Number Theory to a considerable extent, now that, for example, the numbers e and π, have been calculated to over 2000 decimal places with the aid of modern electrical digital computers. Methods are also on the way and plans have been worked out to test numerically, with the same apparatus, the famous Riemann hypothesis which conjectures that all the non-trivial zeros of the function $\zeta(s)$ lie on the line $\mathrm{Re}\, s = \frac{1}{2}$. This hypothesis is of the utmost importance for modern Number Theory and many results in Higher Arithmetic could be considerably refined if the truth of it could be proved. This problem however, is still unsolved today, and so far it has defied the most ingenious, intricate and elaborate methods of modern analysis, even when applied by the greatest among modern mathematicians. Technology may—perhaps very soon—produce some decisive contributions in this most interesting field. It can certainly be said that the whole of Num-

ber Theory would exhibit quite a different aspect if, with modern electrical methods, one single non-trivial zero of $\zeta(s)$, out of the infinite number that we know exist, were to be found elsewhere than on the line $\mathrm{Re}\, s = \frac{1}{2}$.

Harmonious collaboration between pure mathematicians and trained technologists is now beginning to yield far reaching results of fundamental importance in Number Theory, as in so many other branches of science. A case in point is the Mathematical Center in Amsterdam, where the computation department—amply equipped with electrical and mechanical computers—forms a valuable and stimulating support for those engaged in abstract research, both inside and outside this institute.

These are the lines along which, I feel, we may expect great progress in this almost unlimited field, full as it is of the most beautiful relations, absolutely peculiar to it and for which we look in vain in other branches of mathematics—the field of The Theory of Numbers.

REFERENCES

(1) G. H. HARDY, "A Mathematician's Apology," Cambridge, 1948.
(2) EDMUND LANDAU, "Vorlesungen ueber Zahlentheorie," Leipzig, 1927.
(3) H. N. V. TEMPERLEY, "Statistical Mechanics and the Partition of Numbers," *Proc. Cambr. Phil. Soc.*, Vol. 48, p. 683 (1952).
(4) M. L. CARTWRIGHT, "Non-linear Vibrations: a Chapter in Mathematical History," (Presidential address to Mathematical Association), *The Mathematical Gazette*, May 1952, p. 81.
(5) SIR WILLIAM THOMSON, "On the Mutual Attraction or Repulsion Between Two Electrified Spherical Conductors," (1) *Journal de Mathématiques*, 1845, (2) *Phil. Mag.*, April and August, 1853. (Both reprinted in his "Papers on Electrostatics and Magnetism," London, 1872, pp. 86 and 144.)
(6) G. H. HARDY AND E. M. WRIGHT, "The Theory of Numbers," Oxford, 1938, p. 262.
(7) R. FUETER, "Synthetische Zahlentheorie," Berlin, Leipzig, 1921, p. 7.
(8) E. T. WHITTAKER AND G. N. WATSON, "Modern Analysis," Cambridge, 1935, p. 265.
(9) BALTH. VAN DER POL AND H. BREMMER, "Operational Calculus Based on the Two-Sided Laplace Transform," Cambridge, 1950.
(10) J. H. JEANS, "The Mathematical Theory of Electricity and Magnetism," Cambridge, 1915, pp. 196–199.

APPENDIX

The result of the calculation, with the help of Kelvin images, of the capacitance $C_{a,b}$ between two mutually external spheres of radius a and b, respectively, having their centers a distance c apart, can be written in the form:

$$C_{a,b} = \frac{EI}{c} \sum_{n=1}^{\infty} \frac{\alpha^n}{1 - \alpha^{2n}} = \frac{EI}{c} h(\alpha), \qquad \text{say (10)} \tag{1}$$

where

$$\alpha = \frac{E - I}{E + I},$$

while

$$E^2 = c^2 - (a - b)^2$$

and

$$I^2 = c^2 - (a + b)^2,$$

E and I representing the lengths of the external and internal tangents to the two great circles of the spheres in a plane through their centers.

Now, the series in (1) can be transformed as follows:

$$h(\alpha) = \sum_{n=1}^{\infty} \frac{\alpha^n}{1 - \alpha^{2n}} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \alpha^{(2m+1)n}$$

which expression leads at once to the following:

$$h(\alpha) = \sum_{n=1}^{\infty} \left(d(n) - d\left(\frac{n}{2}\right) \right) \alpha^n, \tag{2}$$

where $d(n)$ is the number of divisors of the number n (for example, $d(6) = 4$ because 6 has the four divisors 1, 2, 3 and 6), and where $d\left(\frac{n}{2}\right)$ has to be replaced by zero when n is odd.

The series of (2) is closely related to the famous divisor problem of Dirichlet (Ref. 6, p. 262).

In order to derive an approximation for (2) for the case where the distance between the spheres is small compared with their radii (corresponding to α being just below unity), we can proceed as follows.

We introduce the new variable x through the relation

$$\alpha = e^{-e^{-x}}$$

so that (2) becomes

$$h(e^{-e^{-x}}) = \sum_{n=1}^{\infty} \left\{ d(n) - d\left(\frac{n}{2}\right) \right\} e^{-ne^{-x}} = \sum_{n=1}^{\infty} d(n) \{ e^{-ne^{-x}} - e^{-2ne^{-x}} \}. \tag{3}$$

The operational image (9) of $e^{-e^{-x}}$ is $\Pi(p)$ or

$$e^{-e^{-x}} \doteqdot \Pi(p), \qquad (\mathrm{Re}\, p > 0)$$

so that we obtain

$$h(e^{-e^{-x}}) \doteq \left(1 - \frac{1}{2^p}\right) \Pi(p) \cdot \zeta^2(p). \qquad (\mathrm{Re}\, p > 1)$$

An approximation of (3) for $x \to \infty$, is given by the residue at $p = 1$ of the inversion integral

$$h(e^{-e^{-x}}) \approx \frac{1}{2\pi i} \oint_{p=1} \left(1 - \frac{1}{2^p}\right) \frac{\Pi(p)}{p} \zeta^2(p) e^{px} dp$$

$$= \frac{1}{2\pi i} \oint_{p=1} \left(1 - \frac{1}{2^p}\right) \frac{\Pi(p)}{p} \left(\frac{1}{p-1} + \gamma + \cdots\right) \times \left(\frac{1}{p-1} + \gamma + \cdots\right) e^{px} dp$$

(where $\gamma = 0.57722\cdots$ is Euler's constant), or, with $p = 1 + z$,

$$= \frac{e^x}{2\pi i} \oint_{z=0} \left(1 - \frac{1}{2^{z+1}}\right) \Pi(z) \left(\frac{1}{z} + \gamma + \cdots\right)\left(\frac{1}{z} + \gamma + \cdots\right) e^{xz} dz$$

$$= \frac{e^x}{2\pi i} \oint_{z=0} \tfrac{1}{2}(1 + z \log 2 + \cdots)(1 - \gamma z + \cdots) \left(\frac{1}{z^2} + \frac{2\gamma}{z} + \gamma^2\right) e^{xz} dz.$$

Hence, retaining in the integrand only the terms with z^{-1} we obtain:

$$h(e^{-e^{-x}}) \approx \frac{e^x}{2\pi i} \oint_{z=0} \frac{1}{2z} (\gamma + x + \log 2) dz = \frac{e^x}{2} (\gamma + x + \log 2), \quad (4)$$

which is the required asymptotic approximation for (3) when $x \to \infty$. Returning to the original geometrical variables where now $I \ll E$, we obtain:

$$e^{-e^{-x}} = \frac{E - I}{E + I} \approx 1 - \frac{2I}{E};$$

hence

$$e^{-x} = -\log\left(1 - \frac{2I}{E}\right) \approx \frac{2I}{E}, \qquad \text{or} \qquad x \approx \log \frac{E}{2I}.$$

Substitution in (4) yields

$$h(\alpha) \approx \frac{1}{2} \cdot \frac{E}{2I} \left(\gamma + \log \frac{E}{2I} + \log 2\right) = \frac{E}{4I} \left(\gamma + \log \frac{E}{I}\right);$$

hence we find as asymptotic expression for the limiting case where $E \gg I$ (spheres very close to each other)

$$C_{a,b} \approx \frac{E^2}{4c} \left(\gamma + \log \frac{E}{I}\right). \qquad (5)$$

This expression can be simplified still further if we introduce the small distance d between the spheres, where

$$d = c - (a + b).$$

Herewith we have approximately

$$E^2 \approx 4ab,$$
$$I^2 \approx 2d(a + b),$$

so that

$$C_{a,b} \approx \frac{ab}{a+b}\left(\gamma + \tfrac{1}{2}\log\frac{2ab}{d(a+b)}\right). \tag{6}$$

Again, if we assume the two spheres to be of equal size ($a = b$), (6) becomes

$$C_{a,a} \approx \frac{a}{2}\left(\gamma + \tfrac{1}{2}\log\frac{a}{d}\right).$$

When the distance d is so small that we can even neglect γ with respect to $\frac{1}{2}\log\frac{a}{d}$, our final expression for $C_{a,a}$, the capacitance between two spheres of equal radius a, which are a very small distance d apart, becomes

$$C_{a,a} \approx \tfrac{1}{4}a\log\left(\frac{a}{d}\right).$$

Conidendrin-Derived Compounds Tested as Stabilizers.—Two new antioxidants, *alpha-* and *beta-* conidendrol—polyphenolic compounds derived from conidendrin, a substance occurring in many coniferous woods—have been compared with accepted stabilizers in a variety of products by chemists of the Bureau of Agricultural and Industrial Chemistry.

Members of the Oilseed Division of the Southern Regional Research Laboratory in New Orleans, Louisiana, tested the conidendrols as inhibitors of oxidative rancidity in cottonseed and peanut oils, in lard, and in fat-containing candies, as stabilizers of synthetic rubber against aging and hardening, and as inhibitors of polymerization of certain vinyl-type monomers. They found that the conidendrols performed as well as or better than a number of the accepted stabilizers, including some marketed at higher prices than would be the conidendrols.

Since conidendrin, the parent material, is plentiful as a waste product during the pulping of western hemlock—an estimated 10 to 15 pounds can be recovered from the pulping of 1 ton—the derivatives would be of interest as potential replacements or supplements for stabilizers obtained from coal tar (benzene or naphthalene), if these became scarce.

Before the conidendrols can be considered for use in stabilizing food products, toxicity studies now under way at the Western Regional Research Laboratory of the Department of Agriculture must be completed.

Plastic Photographic Filters.—A new plastic filter of high optical quality, designed specifically for use with du Pont "Varigam" variable contrast photographic paper, has been developed after several years of research and evaluation. The plastic filters are expected to supersede the lacquer-protected gelatin filters currently in use. In addition to being flat and of good optical clarity, the new filters are durable and have surfaces highly resistant to scratching and to moisture. They have excellent color stability and are low in cost.

Mounted plastic filters are supplied in one size only and consist of a black plastic frame 2½ in. square with a 2⅛-in. circular filter aperture. Sets of five or ten filters are packaged in an attractive blue and yellow plastic box which serves as a handy storage file for the filters. A molded black plastic filter holder, which fastens securely to the enlarger lens barrel, has also been designed for the new filter sets. Included with each set is a filter selection chart.

Plastic "Varigam" filters in unmounted sheet form are available on special order. These are intended for use in enlargers between the light source and negative, in contact printing, and for other special adaptations.

By use of these specially prepared filters, which range in color from blue to yellow, "Varigam" paper can be made to produce a normal print from any negative ranging in contrast from soft to hard. Conversely it can produce 10 degrees of contrast from the same negative. The paper is coated with a unique emulsion, which gives a print of high contrast when exposed to blue light and a print of soft gradation when exposed to yellow light, while any intermediate degree of contrast can be obtained by using a predetermined mixture of blue and yellow light as provided by any one of the graded filters.

OPERATING MODES OF A SERVOMECHANISM WITH NON-LINEAR FRICTION

BY

HENRI LAUER [1]

SUMMARY

This paper describes the effects of non-linear friction on the operation of a simple positioning servomechanism. These effects are studied by phase-plane representation methods and Liénard's geometrical construction of integral curves. The friction characteristic of the system can be used directly for this construction by suitably choosing the space and time units and coordinates.

Among the conclusions reached, it is found that the steady-state operating conditions depend not so much on the actual speed value as (*a*) on the portion of the characteristic over which the system operates, and (*b*) on the manner (direction and rate) in which the operating point of the system is approached. The operation may, accordingly, be stable (quiescent), unstable (oscillating), or conditionally stable, depending on the form of the applied input function.

INTRODUCTION

The positioning servomechanism discussed is illustrated in the block diagram, Fig. 1. The respective positions of the input and output

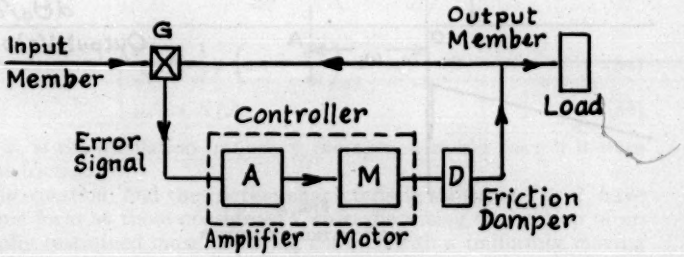


FIG. 1.

members are compared by a differential device *G*. For any misalignment of the two members this device develops a proportional "error" signal. This signal is then fed into a controller which includes an amplifier *A* and motor *M;* the latter, finally, drives the output member and its connected load in a direction tending to nullify the error. The friction of the system is represented, in the diagram, by a damper *D*.

In a servomechanism with viscous output damping the retarding friction torque is proportional to the output velocity. However, when non-linear friction (static or Coulomb friction) is present, the friction

[1] Radio Corporation of America, RCA Victor Division, Camden, N. J.

torque becomes a complicated function of velocity; rather than attempting to represent this analytically, it will here be represented by a curve. The friction characteristic considered is illustrated in Fig. 2, in which the horizontal axis shows the output member velocity, while the vertical axis shows the torque necessary to overcome the friction torque in order to maintain the motion of the output member. The output member being at rest, a torque greater than S must be applied to it to set it in motion. Once the output member is moving, the torque at first decreases from S to F, and then increases along the portion FT of the characteristic as the speed becomes greater. The decrease from S to F is here shown as a sudden drop. In a later section of this paper a less abrupt reduction of the friction torque will be considered, which alters somewhat the conclusions obtained with the characteristic shown in Fig. 2.

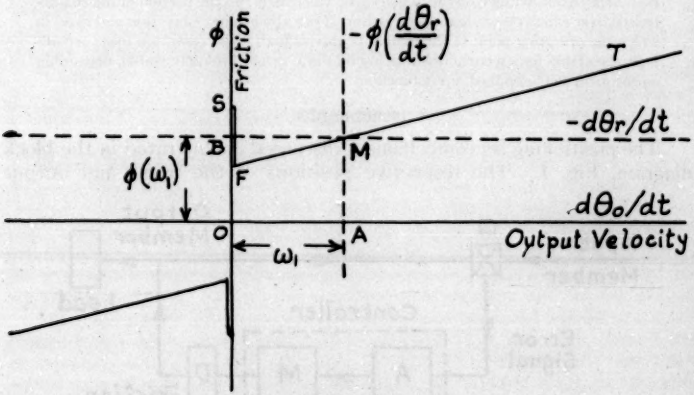


FIG. 2.

THE EQUATION OF MOTION

The following symbols are used in the present discussion:

θ_i, position of input member.

θ_o, position of output member.

$\theta = \theta_i - \theta_o$, position error between input and output members.

K, controller constant (controller output torque per unit error).

J, moment of inertia of the system, referred to the output member.

ϕ, output friction torque. Since its value is a function of the output velocity $d\theta_o/dt$ it will, as necessary, be designated by $\phi(d\theta_o/dt)$ or some equivalent expression. This should not be understood as a product of ϕ by $d\theta_o/dt$ but as an indication, in parentheses, of the particular output velocity value for which the friction torque ϕ is considered.

Other symbols will be defined as they appear in the text.

As for a viscous damped servo, the equation of motion is obtained by expressing that the accelerating force (controller output torque $K\theta$) equals the retarding forces (output inertia force $Jd^2\theta_o/dt^2$ and friction torque $\phi(d\theta_o/dt)$):

$$K\theta = J\frac{d^2\theta_o}{dt^2} + \phi\left(\frac{d\theta_o}{dt}\right). \tag{1}$$

Since by definition

$$\theta = \theta_i - \theta_o \tag{2}$$

Eq. 1 can be re-written in terms of input and error angles and their time derivatives, and becomes

$$K\theta = J\frac{d^2\theta_i}{dt^2} - J\frac{d\theta}{dt} + \phi\left(\frac{d\theta_i}{dt} - \frac{d\theta}{dt}\right). \tag{3}$$

For a step velocity input whereby the input member, originally at rest, is suddenly driven at some constant velocity ω_1

$$\begin{cases} \theta_i = 0, & t < 0 \\ \theta_i = \omega_1 t, & t > 0 \end{cases}$$

Eq. 3 becomes

$$K\theta = -J\frac{d^2\theta}{dt^2} + \phi\left(\omega_1 - \frac{d\theta}{dt}\right) \tag{4}$$

or

$$\begin{cases} \dfrac{d^2\theta}{dt^2} - \dfrac{1}{J}\phi\left(\omega_1 - \dfrac{d\theta}{dt}\right) + \omega_n^2\theta = 0 & (5a) \\ \omega_n^2 \equiv K/J, & (5b) \end{cases}$$

where ω_n is the oscillation frequency the system would have if it were without friction.

This equation, and the friction characteristic shown in Fig. 2, have the same form as those considered [2] when discussing the motion of an elastically restrained mass in sliding contact with a uniformly moving belt. In fact, this latter system may be taken as an analog of the servomechanism considered here. The usual mathematical handling will therefore be applied, but in terms of servo parameters and showing the existence of several possible operating modes of the system.

SOLUTION OF EQUATION OF MOTION

Equations 5 will be solved by Liénard's graphical method.[3] This is particularly simple since it makes direct use of the friction-velocity

[2] For example, see A. A. ANDRONOV AND C. E. CHAIKIN, "Theory of Oscillations," Princeton, Princeton University Press, 1949; and J. J. STOKER, "Nonlinear Vibrations," New York, Interscience Publishers, 1950.

[3] A. LIÉNARD, "Etude des Oscillations Entretenues," *Revue Générale de l'Electricité*, Vol. 33, pp. 946–954 (1928).

characteristic curve of Fig. 2. But because it is a geometrical construction it requires the curve to be referred to a suitably transformed system of coordinates and employing a suitable system of units.

The first step in the coordinate-system transformation consists in shifting the origin of coordinates to a point of the characteristic curve which corresponds to the average velocity of the output member. Since the step velocity applied to the input member imparts to the latter a constant velocity ω_1, the output member will follow at this same average velocity, irrespective of such oscillations as may accompany its motion. This determines a point M on the curve, of abcissa $OA = \omega_1$ and ordinate $OB = \phi(\omega_1)$ on the original axes of Fig. 2.

In view of Eq. 2 the output velocity may be expressed in terms of the input velocity and the time rate of change of the error:

$$\frac{d\theta_o}{dt} = \frac{d\theta_i}{dt} - \frac{d\theta}{dt} = \omega_1 - \frac{d\theta}{dt} \tag{6}$$

so that, with point M as the origin of coordinates, the output velocity is expressed

$$\frac{d\theta_o}{dt} = \left(\omega_1 - \frac{d\theta}{dt}\right) - \omega_1 = -\frac{d\theta}{dt}. \tag{7}$$

Similarly, with point M as the origin, the friction torque is expressed

$$\phi\left(\frac{d\theta_o}{dt}\right) = \phi\left(\omega_1 - \frac{d\theta}{dt}\right) - \phi(\omega_1). \tag{8}$$

Finally, it is observed that for an average output velocity ω_1 (in the original coordinate system) the corresponding input-output position error θ_{avg} may be obtained from Eq. 5 by setting the time derivatives as equal to zero:

$$\theta_{avg} = \frac{\phi(\omega_1)}{K}. \tag{9}$$

With point M as the origin, the error differs from θ_{avg} by a "relative error" θ_r defined in terms of the absolute error θ and average error θ_{avg} by the relation

$$\theta_r = \theta - \theta_{avg} = \theta - \frac{\phi(\omega_1)}{K}. \tag{10}$$

Substituting the relations (8) and (10) in the equation of motion (4) or (5) of the system, this becomes

$$J\frac{d^2\theta_r}{dt^2} + \phi(\omega_1) - \phi\left(\omega_1 - \frac{d\theta_r}{dt}\right) + K\theta_r = 0 \tag{11}$$

or

$$J\frac{d^2\theta_r}{dt^2} + \phi_1\left(\frac{d\theta_r}{dt}\right) + K\theta_r = 0 \tag{12a}$$

$$\phi_1\left(\frac{d\theta_r}{dt}\right) \equiv \phi(\omega_1) - \phi\left(\omega_1 - \frac{d\theta_r}{dt}\right). \tag{12b}$$

The above use of the relative error θ_r and friction function ϕ_1 in place of, respectively, the absolute error θ and friction function ϕ represents, in the equation, the shift of the origin of coordinates from point O to point M as previously described and as shown by the dotted-line axes in Fig. 2. Note that the curve, in the new system, is a plot of $-\phi_1$ against $-d\theta_r/dt$. A plot of ϕ_1 against $d\theta_r/dt$ can be obtained by turning the entire diagram 180 degrees.

The second step in the coordinate-system transformation consists in using a new time unit, equal to the natural period $1/\omega_n$ of an "undamped" servo (that is, one with no friction) having the same inertia J and controller constant K as the actual servo system. In terms of the original time t the new "natural time" t_n is then expressed

$$t_n = \omega_n t. \tag{13}$$

Dividing through by J and K, and in view of (5b), Eq. 12a then becomes

$$\frac{d^2\theta_r}{dt_n^2} + \frac{1}{K}\phi_1\left(\omega_n\frac{d\theta_r}{dt_n}\right) + \theta_r = 0 \tag{14}$$

which is a form suitable for application of Liénard's construction of the integral curves.

Following a usual procedure this equation is written as a first-order differential equation by introducing an auxiliary variable v

$$v = \frac{d\theta_r}{dt_n} \tag{15}$$

from which

$$\frac{dv}{d\theta_r} = \frac{\frac{-1}{K}\phi_1(\omega_n v) - \theta_r}{v}. \tag{16}$$

For any constant value of $dv/d\theta_r$ this equation defines an isocline of the integral curves in the v, θ_r plane. That is, for such a constant value the sets of values of v and θ_r which satisfy Eq. 16 define a curve in the v, θ_r plane; and this curve will cut the integral curves at points where the latter have a slope equal to the constant value assigned to $dv/d\theta_r$.

That the v, θ_r plane is the same as the one defined by the ϕ_1, $d\theta_r/dt$ axes shown in dotted lines in Fig. 2 may be seen by observing that the

friction characteristic illustrated in that figure is also the "zero-slope" isocline. For if we set $dv/d\theta_r = 0$ in Eq. 16 we must have

$$\frac{-1}{K}\phi_1(\omega_n v) = \theta_r. \tag{17}$$

Now the values of ϕ_1 (and therefore also of $\frac{1}{K}\phi_1$ expressed in suitable units) are given, in Fig. 2, along the vertical dotted-line axis; and the curve shown is a plot of these values against values of v shown along the horizontal axis. Since, according to Eq. 17, $\frac{-1}{K}\phi_1$ may be replaced by θ_r it is seen that the two coordinate axes of the figure indeed define the v, θ_r plane and that the characteristic curve is the zero-slope isocline of the integral curves of the equation.

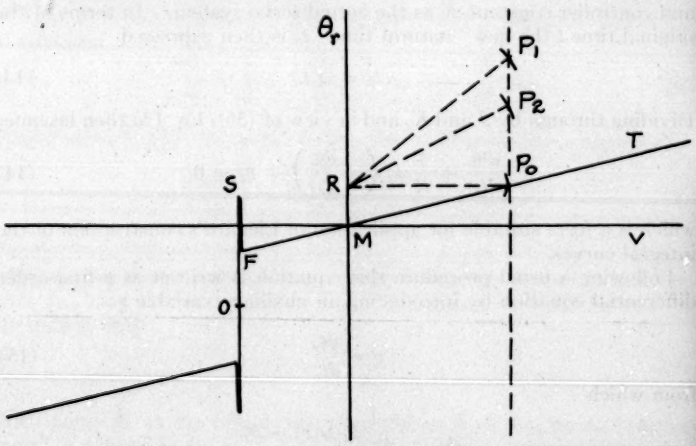


FIG. 3.

Reproducing in Fig. 3 the useful elements of Fig. 2, let $P_1, P_2, \cdots$ represent points of a line parallel to the θ_r-axis which cuts the characteristic curve (or zero-slope isocline) at some point P_0. Let point R be the projection of P_0 on the θ_r-axis. Liénard has shown [3] that the lines $RP_1, RP_2, \cdots$ are normal to the integral curves passing through the points $P_1, P_2, \cdots$. This makes it possible to plot a direction field on the v, θ_r plane showing the slope of the integral curve at every point of

this plane. From this field, integral curves may then be drawn for any particular set of initial conditions.

Examples of such integral curves are discussed in the following paragraphs.

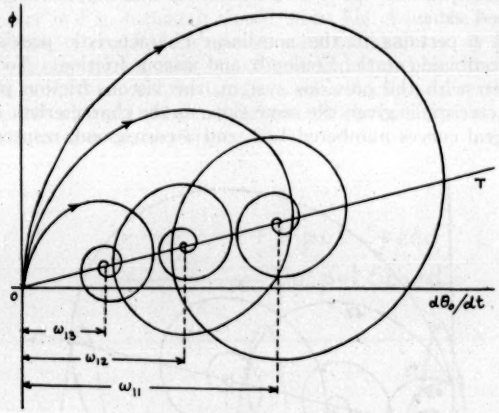


FIG. 4.

UPWARD STEP VELOCITY INPUT OPERATION

Figure 4 corresponds to the case of viscous damping (linear servo system) of less than critical value. The straight line OT is the velocity-friction characteristic. The three integral curves plotted in the θ_r,

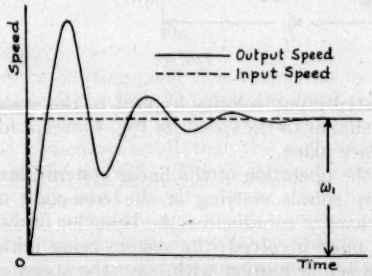


FIG. 5.

$d\theta_r/dt$ plane correspond to three different step velocity inputs. These curves are homologous skew spirals; they start at the origin of the velocity-friction coordinate system and curl, in clockwise direction, around and toward the point corresponding to the applied speed value.

From these integral curves, system performance curves may be obtained, showing the input-output error, the output position, or the output speed as functions of time. As an example of the latter, Fig. 5 illustrates the output speed initial overshoot beyond the input speed, and its subsequent damped oscillation and gradual approach toward the input speed value.

Figure 6 pertains to the non-linear characteristic previously described, combining static, Coulomb, and viscous friction. To facilitate comparison with the previous system, the viscous-friction portion of the characteristic is given the same slope as the characteristic of Fig. 4. The integral curves numbered 1, 2, and 3 correspond, respectively, to

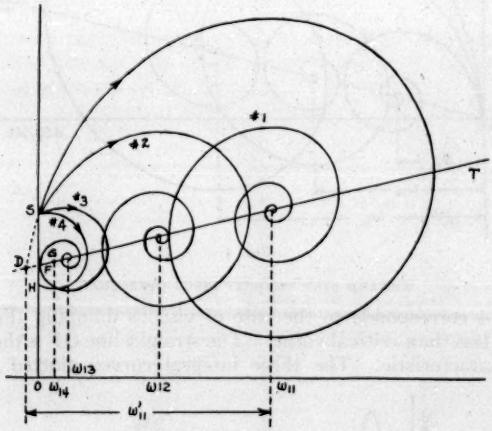


FIG. 6.

three values of step input velocity applied to the system originally at rest. They are similar to the spirals of Fig. 4, except for their positioning in the reference plane.

Thus, while the operation of the linear system illustrated in Fig. 4 is represented by spirals starting at the zero-point of origin of the velocity-friction axes of coordinates, the behavior of the non-linear system of Fig. 6 is more involved: the system being initially at rest, the input member is set in motion with, say, the speed ω_{11}. Because of static friction, the output member at first remains stationary. The error therefore increases from zero at the rate ω_{11} until it reaches the value $\theta_S = OS$ for which the controller torque $K\theta_S$ overcomes the static friction limit $\phi(S)$. During this period, the operating point in the θ_r, $d\theta_r/dt$ plane describes the segment OS in Fig. 6. The output member,

being then suddenly released, accelerates toward the velocity value ω_{11}, overshoots, and finally approaches this velocity value in a damped oscillation as the operating point describes the skew spiral from point S in a clockwise direction.

It should be noted that the overshoot, for a given step input velocity ω_{11}, is greater in Fig. 6 than it would be in Fig. 4 for the same input velocity value. Actually it is the same as would obtain in Fig. 4 for an input velocity shown as ω_{11}' in Fig. 6 and found by extending the spiral from point S toward the left (in counterclockwise direction) to its intersection D with the (similarly extended) viscous-friction portion of the velocity-friction characteristic.

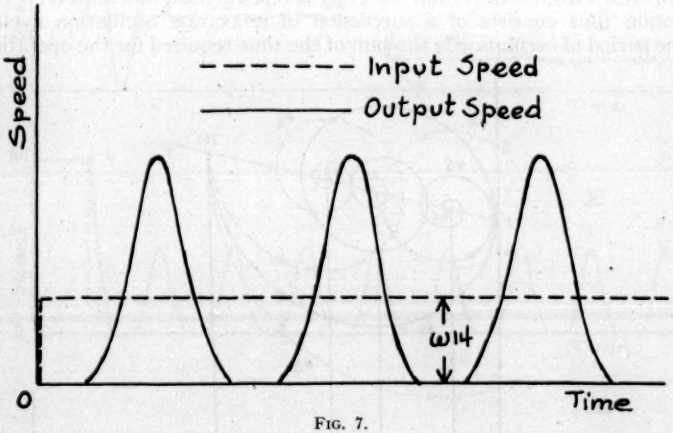


FIG. 7.

Also, because the starting point S of the spirals in Fig. 6 is located outside the viscous-friction line FT while the focal points around and toward which these spirals curl are located on that line, the step input velocity may be chosen so small that the corresponding spiral will become tangent to the line OS. This is shown by the spiral No. 3 in Fig. 6. For any step input velocity *greater* than this limit value ω_{13}, the output member undergoes a damped transient oscillation before reaching a quiescient steady state for which its velocity equals the input velocity. And, as in the case of pure viscous output damping, the oscillation frequency is determined by the (moment of) inertia, controller constant, and viscous friction coefficient of the system.

But for any step input velocity *smaller* than the limit value just defined, the operation of the system assumes an entirely different character, to be described presently.

SMALL UPWARD STEP VELOCITY INPUT OPERATION

Consider the condition illustrated in Fig. 6 when a step velocity ω_{14} smaller than ω_{13} (and corresponding to point G on the friction line FT) is applied to the system originally at rest. As before, the operating point at first describes the static-friction line segment OS. The output member then begins to move, and the operating point follows the spiral path No. 4 (curling around point G) in clockwise direction down to point H, where it encounters the static-friction line OS.

The output member then stops, and the operating point describes the straight line segment HS. Upon reaching the position S the output member again sets into motion, the operating point follows the spiral path No. 4 from S to H, and the process repeats itself indefinitely. The motion thus consists of a succession of relaxation oscillation cycles. The period of oscillation is the sum of the time required for the operating

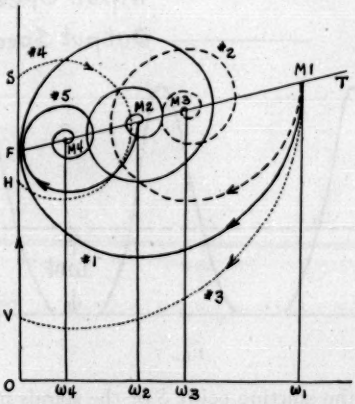


FIG. 8.

point to move along the straight line segment HS (which is proportional to the input speed ω_{14}) and that required for the operating point to follow the portion of spiral path from S to H (which is determined by the inertia, controller factor, and viscous friction coefficient of the system). This is illustrated by the curves of Fig. 7, which show the input and output speeds against time.

DOWNWARD STEP VELOCITY INPUT OPERATION

It was shown in the preceding section that, with the system originally at rest, a step input velocity will cause stable or unstable operation, depending respectively on whether the magnitude of the input step is large or small.

However, this should not be taken to imply that low-speed operation is inherently unstable (oscillatory). In fact, stable operation can be obtained at as low a speed as may be desired (for the friction characteristic here shown) by first driving the system at high speed and then slowing it down at a rate smaller than a critical value determined by the friction characteristic.

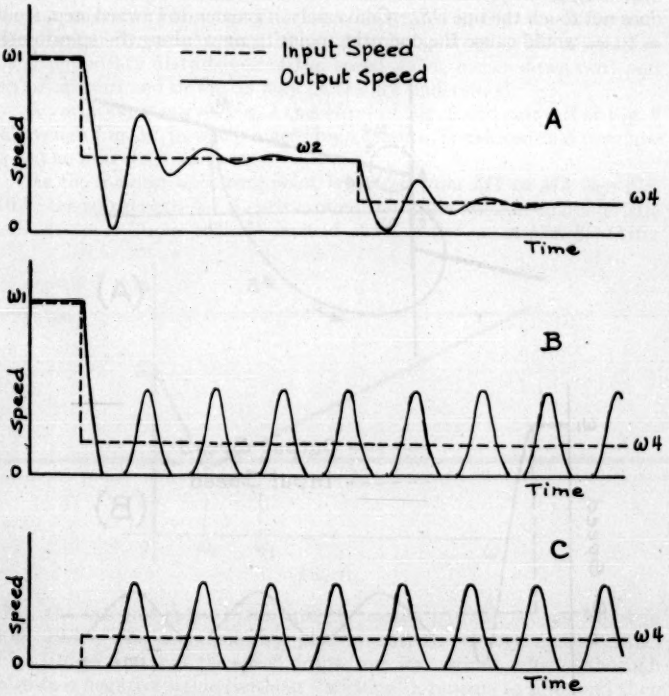


FIG. 9. Operation of non-linear servo at a given speed, ω_4.

(*A*) Stable operation at speed ω_1 changed to stable operation at speed ω_4 by two small downward steps of input velocity $\omega_1 - \omega_2$ and $\omega_3 - \omega_4$ (curves No. 1 and No. 5 of Fig. 8).
(*B*) Stable operation at speed ω_1 changed to unstable operation at speed ω_4 by a single large downward step of input velocity $\omega_1 - \omega_4$ (curves No. 3 and No. 4 of Fig. 8).
(*C*) Unstable operation at speed ω_4 resulting from step input velocity ω_4 applied to system originally at rest (curve No. 4 of Fig. 6).

Figure 8 illustrates the existence of such a maximum rate of decrease of the operating speed, above which "sticking" or relaxation oscillations will occur. The system is first driven at some speed ω_1 corresponding to point $M1$ in the figure. Let then the speed be reduced to a value ω_2 by a downward input step $\omega_1 - \omega_2$: The operating point will move from

$M1$ to $M2$ by following the spiral path No. 1. In the example shown, and for the particular slope of the friction line FT, this speed reduction of about 55 per cent of the original speed ω_1 produces a spiral path that is just tangent to the static friction line OS.

A smaller downward step, from ω_1 to ω_3, would have caused the operating point to move from $M1$ to $M3$ along the path No. 2, which does not touch the line OS. Conversely, a greater downward step, from ω_1 to ω_4, would cause the operating point to move along the spiral path

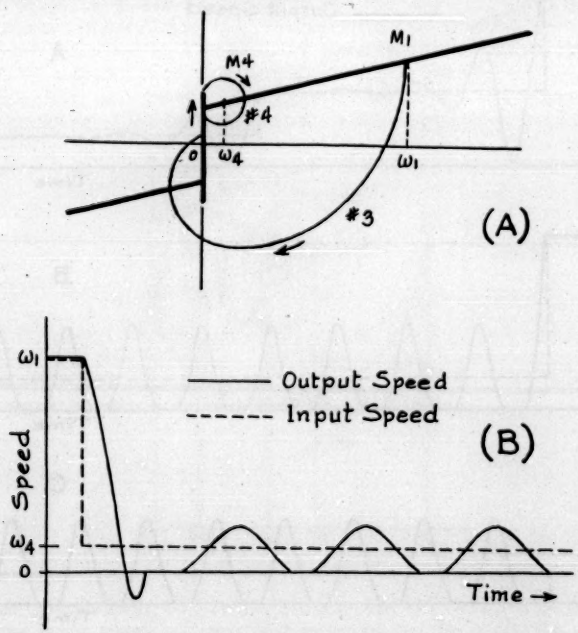


FIG. 10.

No. 3 which encounters the static friction line OS at V. From there, the point moves up along the straight line VS while the output member remains stationary ("sticking"). Upon reaching the position S, the operating point describes the spiral No. 4 and, if ω_4 is sufficiently small, a relaxation cycle HSH is obtained, as described in the previous section, encircling point $M4$ without approaching it.

On the other hand, the speed ω_1 having been reduced to ω_2 by a first downward step, a second step $\omega_2 - \omega_4$, again no greater than 55 per cent

(in our example) of ω_2, will cause the operating point to follow the spiral path No. 5; and this will curl around and tend toward the stable operating point $M4$.

Thus, the operating speed ω_4 may correspond to stable operation as just described, or to unstable operation if approached too rapidly from a high-speed motion condition, or by an upward step applied to the system at rest. These conditions are illustrated in Fig. 9. Moreover, operation at the speed ω_4 may be made to change from stable to unstable by a temporary disturbance of the speed value, either downward and up or upward and down, as may be readily understood.

A variation of curves No. 3 and No. 4 of Fig. 8 and curve B of Fig. 9 is given in Fig. 10, in which a very high speed ω_1 is reduced to a very low speed ω_4 in a single large step.

As the average operating point is shifted from $M1$ to $M4$ (see Fig. 10A) the spiral path No. 3 curls around the static-friction characteristic and strikes it "from behind" instead of "directly" as in Fig. 8. After

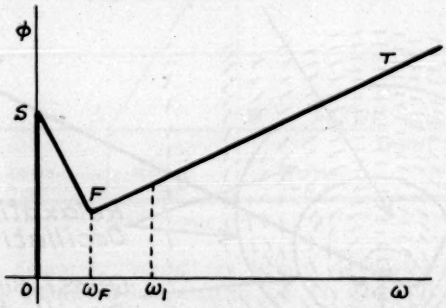


FIG. 11.

this, the integral curve turns into the closed path shown as No. 4 in Fig. 10A. The corresponding plot of output velocity against time, Fig. 10B, shows that the speed drops from its original value ω_1 through zero to a negative value (without "sticking"), returns to zero, and then describes a relaxation oscillation with periodic intervals of "sticking" alternating with cycle portions of free oscillation.

GRADUAL FRICTION CHARACTERISTIC

The friction characteristic shown in Fig. 11 differs from the one considered previously in that the friction force drops from S to F along a line segment SF of finite negative slope instead of falling along a portion of the vertical friction-torque axis. In other words, the point F of minimum friction corresponds to a finite value ω_F of the average driving speed.

By plotting a direction field of the integral curves, the following conclusions are reached:

1. The system being originally at rest, a step velocity input of large value causes the output member to undergo a damped transient oscillation and then to assume a constant velocity equal to the input velocity.

2. On the other hand, a very low value of input velocity (less than that corresponding to point F of the characteristic) always gives rise to a state of continuous oscillation. Depending on particular circumstances, this may be of either quasi-sinusoidal or non-sinusoidal waveshape.

3. Finally, for an input velocity but slightly larger than that corresponding to point F of the characteristic, the direction field of the integral curves is as shown in Fig. 12. It contains a closed path en-

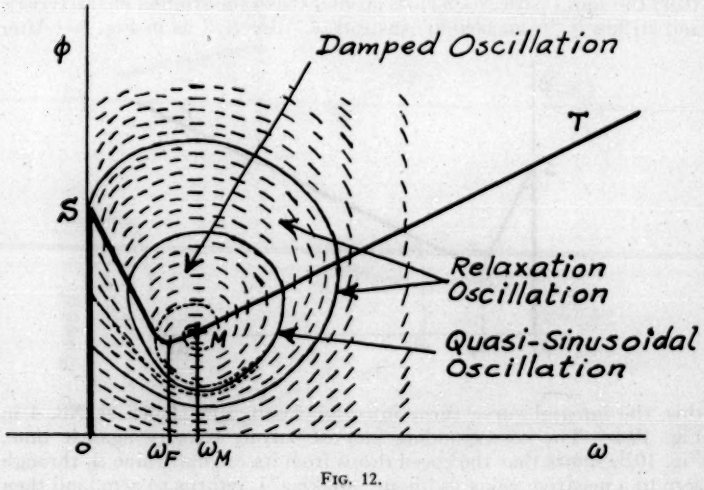


FIG. 12.

closing the average operating point M. This path represents a continuous, quasi-sinusoidal oscillation of the output member which, however, is an unstable condition. This is shown by the fact that the direction field of the integral curves at points located inside the closed path converge toward the average-speed (input-speed) point M, indicating a damped transient oscillation; while at points outside the closed path, the integral curves diverge until they encounter the vertical axis of coordinates, giving rise to a relaxation oscillation (with periodic "sticking") of the type previously studied.

The average "diameter" of the closed path just described under (3) is zero when the input speed has the value corresponding to point F of the characteristic. From this value up, it increases with the input speed until the closed path passes through point S on the vertical axis. Above this input speed value, the closed path disappears and the output member either goes through a sustained relaxation oscillation or stabilizes in a constant-speed motion after a damped transient oscillation.

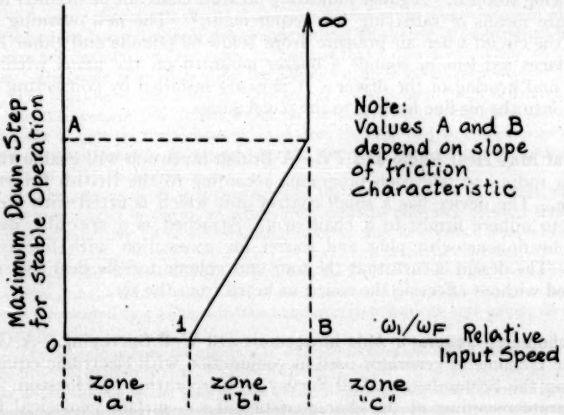


FIG. 13.

Thus, three operating "zones" are defined, as illustrated in Fig. 13 in terms of the quantities indicated in Fig. 11, which identify certain portions of the friction characteristic:

Zone a, oscillating stability of the system, within which the system performs steady oscillations.

Zone b, dual or conditional stability of the system, whereby quiescent operation may be reached on a "downstep," and relaxation oscillations are obtained on an "upstep."

Zone c, stable quiescent operation of the system.

Air Pressure Warning Switch.—In order to comply with a regulation by the Interstate Highway Commission, to become effective June 30, 1953, the Stewart-Warner Corporation has developed an air pressure warning switch of the type that will become mandatory equipment on every bus, truck and truck-tractor equipped with air brake systems. The regulation provides that "every truck, bus and truck-tractor equipped with an air brake system shall be provided with either an audible or visible warning signal to indicate readily to the driver any loss or lack of air sufficient to prevent the vehicle from being stopped. A gauge indicating pressure shall not be deemed to be an adequate means of satisfying this requirement." The new warning switch closes the circuit when air pressure drops below 60 pounds and either lights a multi-facet red lens or sounds a buzzer mounted on the panel within easy vision and hearing of the driver. It is easily installed by connecting a "T" fitting into the air line leading to the panel gauge.

Deaf May Hear Radio and TV.—A British invention will enable the deaf to hear radio and television programs, according to the British Information Service. The device has a small control unit which is fitted with weighted fabric to adhere firmly to a chair arm. Attached is a specially designed single headphone with plug and socket for connection with television or radio. The design is such that the tone and volume for the deaf user can be adjusted without affecting the sound as heard from the set.

Tachometer Generator Aids in Accurate Oil Well Surveying.—A General Electric tachometer generator used in conjunction with electronic equipment is aiding the Schlumberger Well Surveying Corporation of Houston, Texas, in accurate recording of the characteristics of sub-surface geological formations encountered in oil well drilling.

The instrument enables Schlumberger well surveyors to read and maintain a more uniform speed while lowering or withdrawing instrument bearing cables in the drill holes.

The many services of the corporation include the recording of electrical resistivity, potential, and radioactive characteristics of sub-surface formations. These measurements are presented in a continuous recording made while the instruments are moved along the bore hole—often at a depth of two or three miles below the surface. Interpretation of the resultant logs reveals much sub-surface information.

It is important in the operation of these surveys to maintain a fixed and constant speed. To utilize the General Electric tachometer, Schlumberger engineers pass the test cable over a wheel located at the rear of the survey truck. The shaft leading to the tachometer generator from the wheel imposes a motivating force on the generator which causes it to produce a voltage proportional to the speed of the cable passing over it. This voltage is then directed to the tachometer which interpolates it in feet per hour.

Experienced operators know that mechanical measurement is not accurate. According to Schlumberger engineers, the adaptation of the G-E tachometer equipment to this kind of survey has solved their problem of maintaining the necessary accuracy of speed control.

FUNDAMENTALS IN THE BEHAVIOR OF ELECTRETS *

BY

W. F. G. SWANN [1]

ABSTRACT

The paper cites certain results of an earlier paper [2] by the present writer in which are traced the consequences of the assumption that an electret is characterized by a state of polarization, semi-permanent in the sense that it decays very slowly with the time. The decay takes place under the influence of an ohmic conductivity.

The behavior of the dielectric in the case of an exponential decay is determined by the initial polarization P_0, which may vary across the electret, the initial potential difference V_0 between the surfaces of the electret, the coefficient, α, in the exponential decay, and the quantity RC representing the product of the resistance and capacity associated with the electret.

In the present paper it is demonstrated that, in the case of an electret which shows a maximum in the curve representing potential difference between surfaces plotted against time, all of the foregoing quantities may be obtained from measurements of the initial potential difference, the maximum potential difference, the time for the potential difference to fall to zero, and the time for it to attain its maximum. The theory of the matter is extended to the case where the semi-permanent polarization is represented by a finite or infinite number of contributions, each having its own particular relaxation time.

The paper discusses the question of what determines the initial potential difference after the electret is removed from its forming plates. It is shown that a reasonable understanding of the facts can be obtained qualitatively and quantitatively by assuming that on separating the plates slightly from the surfaces of the electret the charge densities retained by the plates are limited by the condition that the resulting field in the air space is less than the breakdown field in air, at the moment of separation.

INTRODUCTION

It will be recalled that a typical procedure for producing an electret is the following: Two plates, A and B, are supported in a suitable molten dielectric, for example, carnauba wax. A is at a potential V_0*, which we shall consider positive and which is of the order of 6000 volts. Plate B is at zero potential. The potential difference is maintained while the wax cools and solidifies, and the plates are then removed, so that the piece of wax which was between them is now isolated except that the plate B, maintained at zero potential, may be left attached if so desired, Fig. 1. Under these conditions, the upper surface of the

* A paper presented on January 23, 1953, at the New York Meeting of the American Institute of Electrical Engineers.

[1] Director, Bartol Research Foundation of The Franklin Institute, Swarthmore, Pa.; and Senior Advisor, The Franklin Institute Laboratories for Research and Development, Philadelphia, Pa.

[2] W. F. G. SWANN, "On Certain Matters Pertaining to Electrets," JOUR. FRANKLIN INST., Vol. 250, p. 219 (1950); see also, W. F. G. SWANN, "Electrets," *Phys. Rev.*, Vol. 78, p. 811 (1950).

dielectric will be found at a potential different from zero, usually *negative* and of the order of 2000 or 3000 volts. We shall denote by a the surface of the dielectric which was in contact with A during formation of the electret.

One can now perform experiments which are the equivalent of measuring the potential of surface a as a function of time, and starting at some arbitrary negative potential of -3000 volts, one obtains a curve of the general nature of Fig. 2. After a rise to zero potential in two or three days or less, the potential continues to rise in the positive sense for about two weeks. After reaching a positive maximum of perhaps 2000 volts, it proceeds to fall slowly, in a manner which eventually

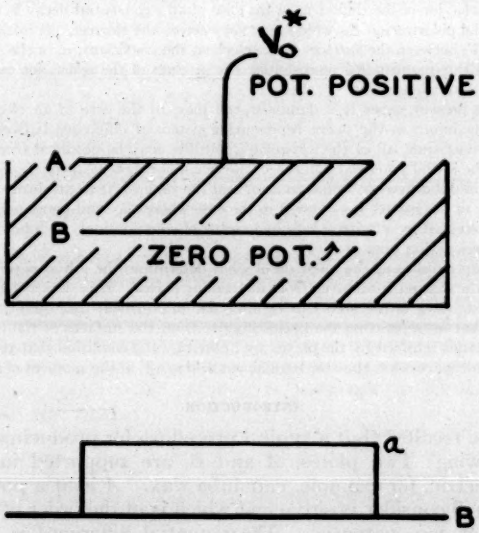


FIG. 1.

approximates something like an exponential form, and at such a rate that a year or more may elapse before it has fallen to half of its maximum value.

If, during the development of such a curve as Fig. 2, one should bring the surface a to zero potential, that is, to the potential of plate B, for example, by flaming the surface, and if the electret be then allowed to follow its natural course, the surface a will once more proceed to rise to a positive maximum and then decay to zero over a long period of time.

It has long been recognized that the essential phenomena can be

accounted for by the assumption that the formation of the electret is associated with the creation of a polarization which is in the direction of the activating field and which, when the electret has cooled under the influence of the field, is semi-permanent in the sense that it decays very slowly with the time.

In a comparatively recent paper,[2] the present writer has developed the mathematical consequence of the foregoing hypothesis and has shown that the characteristic phenomena of electrets can be accounted for qualitatively and quantitatively in terms of it.

GENERAL PICTURE OF THE STATE OF AFFAIRS IN AN ELECTRET IMMEDIATELY SUBSEQUENT TO FORMATION

It will be interesting at this stage to consider the picture presented by an electret possessing a semi-permanent polarization and otherwise

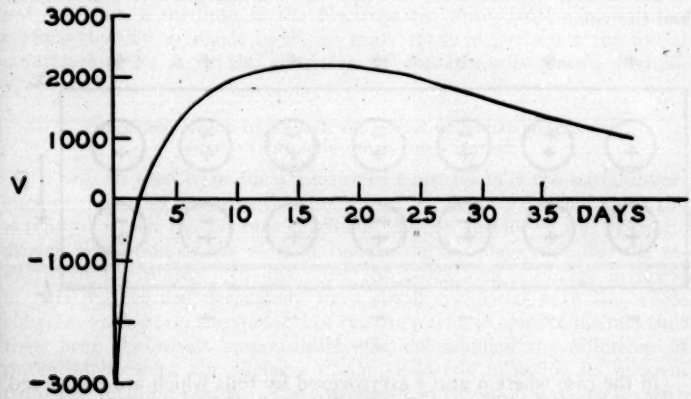


FIG. 2.

subjected to such conditions that its two faces are at the same potential, this electret having been *formed* under conditions in which the plate A adjacent to the surface a was at a positive potential. The situation is represented in Fig. 3, where, on the surface a, there is a positive surface density σ, on the surface b a negative surface density $-\sigma$, and a semi-permanent polarization P between. The general relationship between these quantities is

$$4\pi\sigma = D = 4\pi P, \tag{1}$$

where D is the dielectric displacement.

In the general case, where there is a finite potential difference between a and b, so that a field E exists, there is an induced polarization $(K - 1)E/4\pi$, where K, the specific inductive capacity, is the change

of D per unit change of E. Under these conditions

$$4\pi\sigma = D = E + 4\pi[P + (K - 1)E/4\pi] = KE + 4\pi P. \qquad (2)$$

When E is zero, this reduces to Eq. 1.

In the case where the dielectric is a perfect insulator, if P should change with time, we should have, from (2), since σ remains constant,

$$K(E - E_0) = 4\pi(P_0 - P),$$

where E_0 and P_0 refer to $t = 0$. Thus, even if we should start with zero difference of potential between surfaces a and b, that is, with $E_0 = 0$, a field E, given by

$$KE = 4\pi(P_0 - P)$$

would develop by the time t, when the semi-permanent polarization had the value P.

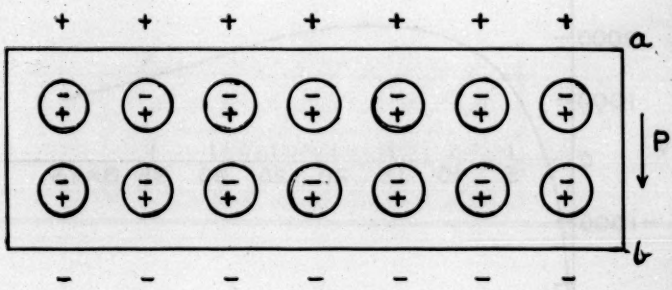


FIG. 3.

In the case where a and b are covered by foils which are connected together and joined by a perfectly conducting wire, the field would be zero at all times and the charge σ would diminish with the time in a manner controlled by (2), with E zero. In other words

$$4\pi\sigma = 4\pi P.$$

By the time P had fallen to zero, σ would have become zero, so that the time integral of the current would be equal to σ_0, and incidentally to P_0.

Now in experiments on electrets the time integral of the current can be measured and it turns out that σ_0 is of the order of 1000 e.s.u., which is a surface density so large that a condenser with plates 1 cm. apart, charged to $+\sigma_0$ and $-\sigma_0$ in a vacuum, would yield a field of the order $4\pi \times 1000$ e.s.u., that is, about 3.6×10^6 volts/cm.

Thus we see that our electret at $t = 0$ with zero potential difference between its surfaces presents a very remarkable situation. We can

regard it as having an enormous surface density of charge such as would yield a field of 3.6×10^6 volts/cm., except that there is also an enormous polarization such as would yield a field of 3.6×10^6 volts/cm. in the opposite direction. A very small percentage alteration in σ_0 would give rise to a large field across the dielectric. Thus, if the positive σ_0 on surface a were reduced by 10 e.s.u., that is, by one per cent, (2) shows that the field in the electret would change by $40\pi/K$ e.s.u., that is, by about 12,000 volts/cm., for $K = 3$, and with surfaces separated by 1 cm. there would be a change of potential difference of 12,000 volts between the surfaces, the surface a being at $-12{,}000$ volts in relation to b, in spite of the fact that it carried an enormous positive surface density of charge. The apparent paradox arises, of course, from the fact that the polarization acts to make a negative to b. It will be recalled that a polarization P in the direction of the outwardly drawn normal from a medium is the electrostatic equivalent of a fictitious surface density σ_f equal to P, so that, since at surface a the initial polarization P_0 is in the direction of the inwardly drawn normal, $\sigma_f = -P_0$.

CONDITIONS WHICH DETERMINE THE INITIAL POTENTIAL DIFFERENCE BETWEEN THE SURFACES OF THE ELECTRET

It will be seen from the above considerations that the initial difference of potential between the surfaces of an electret depends upon the extent to which the charges originally on the surface of the dielectric during formation of the electret remain on the dielectric after the removal of the plates. In any case, the variations of charge involved in this matter are necessarily very small compared with the whole charges retained on the surfaces of the dielectric, in spite of the fact that they bear the whole responsibility for determining the difference of potential between the surfaces of the dielectric following its removal from the forming plates. The variations in question have to do with what we may call the relative "primary affinities" of the plates and dielectric for electric charge. We shall now proceed to discuss this matter.

If the two surface charges were on the dielectric at the time of formation and remained there on removal of the dielectric, the potential of a would be equal to the potential of the plate A during formation of the electret.

It is probable that the actual charge delivered to or carried from the forming plates on removal of those plates is to some extent determined by the nature of the removal. However, we shall study the essentials by considering a simple case where the plates are removed by separating them to a small distance ϵ from the surfaces of the dielectric, keeping them parallel thereto, and then sliding them away. Let us consider the case where, during the formation of the electret, the top plate A

was at a positive potential in relation to B, a potential of, say, +6000 volts. Let us suppose that on sliding the plates away, the top surface a is left with a surface charge density σ_a, and that the bottom surface is caused to rest on a metal plate connected to a conductor of large capacity—the earth for example. Then, whatever may have been the charge density σ_b on the lower surface of the dielectric, the charge density on this plate will adjust itself in relation to what remains on the surface b, so that the two densities together are equal to $-\sigma_a$. Regarding the plate on which the dielectric rests as at the arbitrary zero of potential, the potential of the surface a will now be V_0, given by

$$KV_0 = 4\pi(\sigma_a - P_0)L. \tag{3}$$

Now, suppose that σ_A was the surface density of charge on the plate A just after the plates had been separated from the dielectric by the small amount ϵ. Then, from (2), we have

$$4\pi(\sigma_A + \sigma_a) = \frac{KV_0^*}{L} + 4\pi P_0, \tag{4}$$

where V_0^* was the potential of A in relation to B during formation. Thus, from (3) and (4),

$$KV_0 + 4\pi\sigma_A L = KV_0^*. \tag{5}$$

Now just when the plates are separated from the dielectric by the amount ϵ, and to the approximation implied in the assumption that the distance between the plates is small compared with their linear dimensions, we have no field outside the plates. Thus the field in the gap ϵ will be $4\pi\sigma_A$. Hence, since the breakdown potential of the air in ϵ is 30,000 volts/cm. (=100 e.s.u.) we must have

$$4\pi|\sigma_A| < 100$$
$$4\pi|\sigma_A| = 100\eta,$$

where η can have any value from zero to unity and $|\sigma_A|$ is the absolute value of σ_A.

Thus, from (5)

$$KV_0 = KV_0^* \pm 100\eta L$$

the positive or negative sign depending upon whether σ_A is negative or positive. Thus, if $L = 1$ cm., the possible values of V_0 lie between $V_0^* + \frac{100}{K}\eta$ and $V_0^* - \frac{100}{K}\eta$.

If $V_0^* = 6000$ volts $= 20$ e.s.u., and if $K \sim 3$, V_0 will lie between $(20 + 33)$ e.s.u., and $(20 - 33)$ e.s.u., that is, between 16,000 volts and -3900 volts, the former value corresponding to the extreme possibility

for σ_A negative and the latter to the extreme case where σ_A is positive. The case σ_A positive is perhaps the natural one, since it corresponds to the plate A *holding* some of the positive charge which was on a, when separation occurs, while σ_A negative corresponds to the case where A tended to be negative to the surface of the dielectric with which it was in contact to the extent of actually donating a positive charge to the surface of the dielectric on separation. However, the practical fact is that the surface a seems generally to be at a negative potential after separation, in spite of the fact that it was in contact with the positive plate A during formation. Such a situation corresponds to the initial existence of a so-called "heterocharge." We again emphasize the fact that the percentage of the total charge on a, which has to be left on A in order to cause the surface a to fall from, say, $+6000$ volts to -3900 volts is very small.

In the light of the foregoing view to the effect that the initial potential of a, after removal from the forming plates, is determined by the breakdown field of the air, it would be of interest to separate the dielectric from the plates under high pressure and see how such a procedure would affect the said initial potential. It may be remarked, however, as will presently be shown, that to a first approximation, the magnitude of the positive maximum, when such exists, is independent of the initial potential of a and depends only upon the initial polarization, its decay characteristics, the capacity of the electret, and the specific resistance of the material composing it.

In connection with all of the above, there is every reason to believe that if, in the formation process, the plate A had been negative to the plate B, all of the conclusions reached would still hold with a change of sign of the potentials, except that the amount of charge removed by the plate A on separation of the dielectric from its forming plates would not necessarily be of the same amount with changed sign. In fact, the sign of the charge *removed* might even be unaltered. Such circumstances reflect themselves in the determination of the potential of the surface a immediately after removal of the electret from the forming plates.

HISTORY OF THE ELECTRET SUBSEQUENT TO FORMATION

In the paper already cited,[2] the present writer has traced the detailed behavior of an electret subsequent to its formation on the basis of hypotheses rather more general than those cited above. The surface charge distribution and uniform semi-permanent polarization are generalized to the stage of admitting an initial volume distribution of charge and a semi-permanent polarization, both of which can vary in amount as one passes from one surface to the other. The theory was carried through in this form to a certain stage without the assumption of Ohm's law and, for further generality, an arbitrary resistance R_s was supposed to connect the two surfaces which, for convenience, were sup-

posed covered with conducting foils. Mathematical difficulties prevented the completion of the problem in this form, but on the simplifying assumption that Ohm's law prevails—that is, that the current density is everywhere proportional to the field—it was possible to carry the calculation through even to the extent of admitting the existence of volume charges and polarization which varied across the dielectric. It turned out that the difference of potential between the surfaces, regarded as a function of the time, was independent of the volume distribution of charge, and the differential equation which determined the variation of V, the potential of the upper surface, with the lower surface at zero potential was found to be

$$\frac{dV}{dt} + \frac{V}{RC} + \frac{4\pi}{K}\frac{d}{dt}\int_0^L P dx = 0, \tag{6}$$

where P refers to the semi-permanent polarization, x to the distance measured from the upper surface, and C is the capacity of the condenser formed by the two foils on the surfaces of the dielectric and the dielectric itself. The resistance R is the external resistance and the internal resistance in parallel, so that, if R_d is the internal resistance

$$\frac{1}{R} = \frac{1}{R_d} + \frac{1}{R_e}.$$

The only laws involved in the proof of (6) are

$$D = E + (K - 1)E + 4\pi P \tag{7}$$

$$4\pi\rho = \text{div } D \tag{8}$$

$$4\pi\sigma_1 = D_1; \qquad 4\pi\sigma_2 = D_2. \tag{9}$$

The equation of continuity, *viz:*

$$\text{div } i + \frac{\partial\rho}{\partial t} = 0, \tag{10}$$

where P is the semi-polarization at the time t, ρ is the volume charge density, D the dielectric displacement, σ_1 and σ_2 are the surface charge densities at the upper and lower surfaces, and i is the current density.

Multiply (6) throughout by C. It yields

$$-\frac{CdV}{dt} = \frac{V}{R} + \frac{4\pi C}{K}\frac{d}{dt}\int_0^L P dx \tag{11}$$

which shows that the system acts like a charged condenser which loses its charge by a conduction current V/R and by some other process which is independent of V, which depends only upon the rate of decay

of the semi-permanent polarization and is given by the last term in Eq. 11.

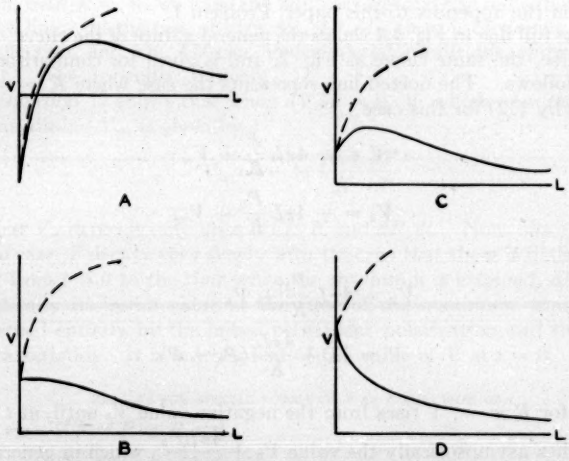
In the case where the semi-permanent polarization is uniform across the electret, (11) becomes [3]

$$-\frac{CdV}{dt} = \frac{V}{R} + \frac{4\pi CL}{K}\left(\frac{dP}{dt}\right)$$

or

$$\frac{dV}{dt} = -\frac{V}{RC} - \frac{4\pi L}{K}\frac{dP}{dt}. \tag{12}$$

The essentials of the story are contained in Eq. 12.



THE DOTTED CURVES RISE SLOWLY AND ASYMPTOTICALLY TO FAR BEYOND THE RANGE OF THE FIGURES.

FIG. 4.

Case where V_0, the Initial Potential of Surface a, is Negative

This represents the normal case. Starting with V_0 at some negative value, both terms on the right hand side make positive contributions so that V rises in potential until it attains the value zero, as in Fig. 2.

When V has attained the value zero, $dV/dt = -\left(\frac{4\pi L}{K}\right)(dP/dt)$

[3] The assumption of uniformity of P is not necessary, as may be seen by reference to the paper already cited. All the formulae here cited are valid for the more general case by replacing P_0L by $\int_0^L P_0 dx$.

and is still positive, so that V rises above zero. Now, however, $-V/RC$ becomes negative and rises in absolute value as V increases, until it eventually cancels the term $-\left(\frac{4\pi L}{K}\right)(dP/dt)$, at which stage dV/dt is zero and we have a maximum. If $-dP/dt$ were constant, the maximum value of V would, of course, be attained only at $t = \infty$ and it would be an asymptotic maximum. If, however, as in the case of exponential decay, $-\left(\frac{4\pi L}{K}\right)(dP/dt)$ diminishes with time, V will attain its maximum at a finite value of t, as in Fig. 2. From here on, V will diminish continually with time to the value zero at $t = \infty$, provided that d^2P/dt^2 is always positive. The proof of this matter is given in the appendix to this paper, Problem 1.

The full line in Fig. 4*A* shows the general nature of the curve. It is, of course, the same curve as Fig. 2, and is given for comparison with what follows. The dotted line represents the case where $R = \infty$. As shown by (12) for this case

$$V = -4\pi L\frac{P}{K} + V_\infty$$

$$V_0 = -4\pi L\frac{P_0}{K} + V_\infty.$$

Hence

$$V_\infty = 4\pi L\frac{P_0}{K} + V_0$$

$$V = V_0 + \frac{4\pi L}{K}(P_0 - P). \tag{13}$$

Thus, for $R = \infty$, V rises from the negative value V_0 until, at $t = \infty$, it reaches asymptotically the value $V_0 + \frac{4\pi LP_0}{K}$, which in general will be positive, since $4\pi LP_0/K$ is large compared with V_0.

Equation 13 applies for $R = \infty$ regardless of the sign of V_0, and always represents an asymptotic rise of total amount $4\pi LP_0/K$ from $t = 0$ to $t = \infty$.

Case Where V_0, the Initial Potential of Surface a, is Positive

Here we have three cases:

Suppose that at $t = 0$, V is positive and such that

$$\frac{V_0}{RC} = -\frac{4\pi L}{K}\left(\frac{dP}{dt}\right)_{t=0}. \tag{14}$$

Then, from (12), $dV/dt = 0$ at $t = 0$. Hence there is a maximum at $t = 0$, and from here on, the story is similar to that already discussed. V diminishes continually to zero at $t = \infty$, as shown by the full curve in Fig. 4*B*. The dotted curve applies to the case $R = \infty$ and represents the asymptotic rise of total amount $4\pi LP_0/K$.

Calling $\bar{V}_0$ the value of V_0 corresponding to (14), suppose that V_0 is positive but less than $\bar{V}_0$. Then, at $t = 0$, dV/dt will be positive, as shown by (12). V will rise to a maximum and then decrease to zero at $t = \infty$, as shown by the full line in Fig. 4*C*. The dotted curve applies for $R = \infty$ and represents the asymptotic rise of total amount $4\pi LP_0/K$.

In the case where V_0 is greater than $\bar{V}_0$, dV/dt will be negative at $t = 0$, and V will decrease monotonically to zero at $t = \infty$, as in Fig. 4*D*. Again, with $R = \infty$, we have the same asymptotic rise indicated by the dotted line. In the case of Figs. 4*A*, 4*B*, 4*C*, there is obviously a point of inflection, and Fig. 4*D* can, under special conditions, show such a point. (See Appendix, Problem 2.)

Equation 12 shows that when $dV/dt = 0$, V, which now represents the maximum V_m, is given by

$$V_m = -4\pi \frac{RCL}{K}\frac{dP}{dt}$$

so that V_m depends only upon RCL/K and dP/dt. Now, since, for the usual case, P decays very slowly with time, so that there is little change in P from $t = 0$ to the time when the maximum is attained, dP/dt has practically its initial value at the time of the maximum, and thus is governed entirely by the initial permanent polarization and the decay characteristics. It is independent of the value of V at $t = 0$.

RESULTS FOR SPECIAL FORMS OF P AS A FUNCTION OF t

The Case Where $P = P_0e^{-\alpha t}$

Here, Eq. 12 assumes the form

$$\frac{dV}{dt} = -\frac{V}{RC} + \frac{4\pi L\alpha P_0 e^{-\alpha t}}{K}$$

which is integrable to the form

$$V = \frac{4\pi RC\alpha}{(1 - RC\alpha)K}\left[e^{-\alpha t} - e^{-t/RC}\right]LP_0 + V_0e^{-t/RC}. \qquad (15)$$

It is of interest to consider the case where V_0 is negative, so that V passes through the value $V = 0$. It is convenient to take a new origin of time, so that $t = 0$ is the instant when $V = 0$. Using τ to denote

time measured from this instant, (15) becomes

$$V = \frac{4\pi RC\alpha}{K(1 - RC\alpha)} \left[e^{-\alpha\tau} - e^{-\tau/RC}\right] LP_0 e^{-\alpha\tau_1}$$

where τ_1 is the interval between $\tau = 0$ and $t = 0$.

We can find the value τ_2 of τ for which $dV/d\tau = 0$, that is, the time for which V attains a maximum. In the paper already cited,[2] it is shown that τ_2 is given by

$$RC\alpha = e^{-\alpha(1/RC\alpha - 1)\tau_2}. \tag{16}$$

The value V_m of the maximum itself is best found by putting $dV/dt = 0$ in (12). This yields

$$V_m = \frac{4\pi}{K}(RC\alpha)LP_0 e^{-\alpha(\tau_1+\tau_2)} = \frac{4\pi RC\alpha LP_0}{K} \tag{17}$$

since $e^{-\alpha(\tau_1+\tau_2)}$ is approximately unity. It is then of interest to find the time for V to change from the value V_0 to zero. τ_1 is obtained by putting $V = 0$ in (15). It is then possible to show [2] that

$$\tau_1 = RC \log_e\left[1 - \frac{V_0}{V_m}\right]. \tag{18}$$

V_0 and V_m can be experimentally measured, so that, by measuring τ_1, we can determine RC, and such a determination is more convenient than one which seeks to measure R and C directly.

By measuring τ_2, the time for V to change from zero to V_m, and with the knowledge of RC obtained from (18), we can now use (16) to determine α.

Finally, knowing $(RC\alpha)$ and K, we can, by measuring V_m, use (17) to determine LP_0 and so P_0. It is thus possible, from measurement of τ_1, τ_2, V_0 and V_m, to determine all the quantities which define the behavior of the electret through (15). It is to be observed that this equation assumes an exponential law of decay for P.

It is convenient to make the following definitions:

$$\tau_e = RC; \qquad \tau_p = 1/\alpha. \tag{19}$$

We may regard τ_e as the electrical time constant and τ_p as the polarization time constant.

It is of interest to reverse the foregoing arguments and trace the consequences as regards τ_1 and τ_2 for a typical case which we shall define in terms of reasonable values. If s is the specific resistance of the material and A the area of the electret surface, then, if there is no

additional resistance across the electret,

$$R = \frac{L}{A} s$$

$$RC = \left(\frac{AK}{4\pi L}\right)\frac{L}{A} s = \frac{K}{4\pi} s$$

so that, since we need only RC and not R and C separately, there is no need to specify A.

Let us then assume as follows:

$$V_0 = -3000 \text{ volts}; \quad V_m = +2000 \text{ volts}$$
$$s = 10^{18} \text{ ohms/cm. for the wax} \sim 10^6 \text{ e.s.u.}$$
$$\alpha = 10^{-8}$$
$$K = 3$$
$$L = 1 \text{ cm.}$$

Then, from (19)

$$\tau_e \equiv RC = 2.2 \times 10^5 \text{ sec.}$$
$$\tau_p \equiv 1/\alpha = 10^8 \text{ sec.}$$

The electrical time constant is thus much shorter than the polarization time constant.

We then find, from (18)

$$\tau_1 = 2 \times 10^5 \text{ sec.}$$

Equation 16 gives $\tau_2 = 13.4 \times 10^5$ sec. and Eq. 17 yields $P_0 = 670$ e.s.u.

Thus, to summarize, the surface a of such an electret, on the foregoing theory, would rise from -3000 volts to zero potential in two days. It would attain a positive maximum in an additional two weeks, and that maximum would be 2000 volts if P_0 were 670 e.s.u. These numerical conclusions are in harmony with the magnitudes observed.

Figure 5 represents V plotted as a function of the time for such an electret as is defined above.

Incidentally, it will be observed that for times large compared with RC, the potential V decays according to the law

$$V = \frac{4\pi RC\alpha LP_0}{(1 - RC\alpha)K} e^{-\alpha t}$$

so that the law of decay becomes determined entirely by α.

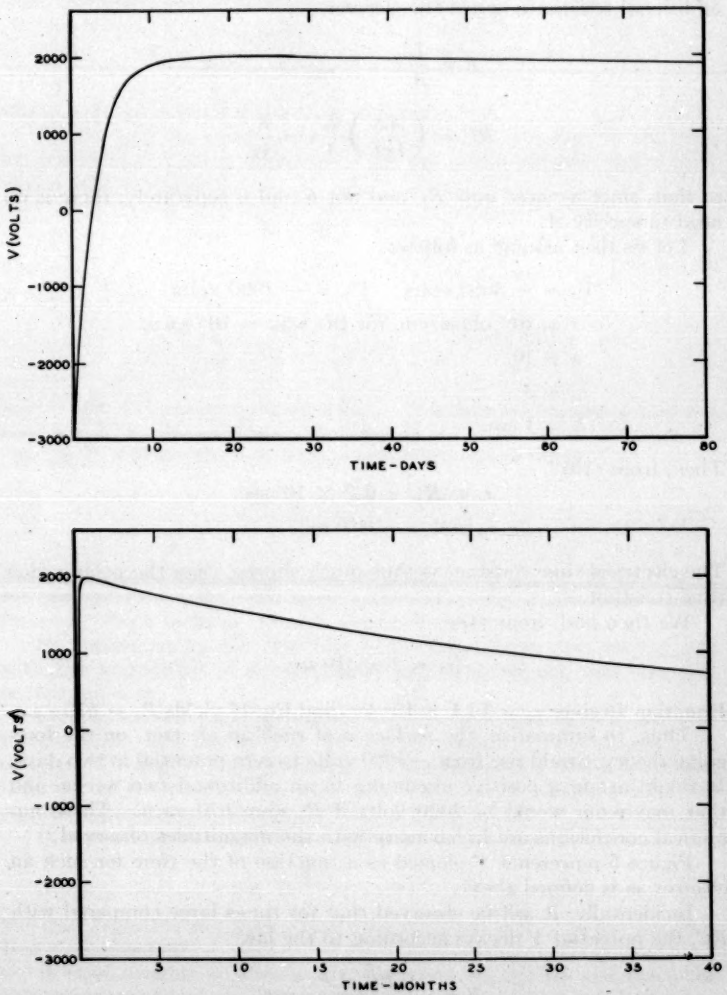


FIG. 5. The two figures correspond to different time scales, the first being given to exhibit more clearly the variation during the first month.

Case Where the Functional Relation Between P and t Is More Complicated Than That of a Pure Exponential

We shall consider the case where P is of the form

$$P = \sum P_m e^{-\alpha_m t}. \tag{20}$$

This type of expression is very general since it corresponds to a number of independent systems, each with its own relaxation time. Of course, in terms of this concept, all the P_m are positive.

Our fundamental differential Eq. 12 for V applies quite generally and in particular for the above form of P, for which case it assumes the form

$$\frac{dV}{dt} + \frac{V}{RC} = \frac{4\pi L}{K} \sum \alpha_m P_m e^{-\alpha_m t}. \tag{21}$$

The solution, subject to $V = V_0$ at $t = 0$ is

$$V = \frac{4\pi L}{K} \sum \frac{RC\alpha_m P_m}{(1 - RC\alpha_m)} [e^{-\alpha_m t} - e^{-t/RC}] + V_0 e^{-t/RC}. \tag{22}$$

If the expression $P = \sum P_m e^{-\alpha_m t}$ is generalized to a continuous distribution of time constants, we have

$$P = \int_0^\infty F(\alpha) e^{-\alpha t} d\alpha. \tag{23}$$

In this case (22) becomes generalized to

$$V = \frac{4\pi L}{K} \int_0^\infty \frac{RC\alpha F(\alpha)}{(1 - RC\alpha)} [e^{-\alpha t} - e^{-t/RC}] d\alpha + V_0 e^{-t/RC}. \tag{24}$$

It is not as easy to utilize V_0, the maximum V_m, and the times for V to reach zero and V_m to evaluate RC, the α_m's and the P_m's of (22) as was the case where only a single exponential was involved in the semi-permanent polarization. However, something can be done. Thus, using (21), and observing that $dV/dt = 0$ for a maximum

$$V_m = \frac{4\pi LRC}{K} \sum \alpha_m P_m e^{-\alpha_m \tau_m} \tag{25}$$

$$\left(\frac{dV}{dt}\right)_{V=0} = \frac{4\pi L}{K} \sum \alpha_m P_m e^{-\alpha_m \tau_1} \tag{26}$$

where τ_m and τ_1 are the times for the maximum and for $V = 0$, respectively. Now to the extent that the significant α_m's are of the order of 10^{-7} or less, the exponential factors will be practically unity at τ_m and

τ_1, so that, approximately, by division of (25) and (26),

$$RC = V_m \Big/ \left(\frac{dV}{dt}\right)_{V=0}$$

and RC is thus determinable in terms of measured quantities. A similar result holds in the case of the assumption (23).

Again, provided that we confine our observations to times large compared with RC, we shall have, in the case of (24)

$$V = \frac{4\pi L}{K}\int_0^\infty \frac{RC\alpha F(\alpha)}{1 - RC\alpha} e^{-\alpha t} d\alpha.$$

By use of the Laplace transformation we can therefore, in principle, obtain $RC\alpha F(\alpha)/(1 - RC\alpha)$ and so $F(\alpha)$ as a function of α if we experimentally determine V as a function of t. Such information gives us a complete determination of the elements which define the problem.

In principle, at any rate, we can regard (22) as a special case of (24). It is perhaps beyond the scope of this paper to discuss the analytical devices which may be utilized to obtain the P_m and the corresponding α_m from experimental observations of V used in conjunction with an assumption of the form (20) based upon a finite number of terms or upon an infinite series.

APPENDIX

PROBLEM I. *Proof that if d^2P/dt^2 is always positive, there would be only one maximum and no minimum for the case where V_0 is negative, and that V, after attaining its maximum, will diminish monotonically to zero at $t = \infty$.*

From (12)

$$\frac{d^2V}{dt^2} = -\frac{1}{RC}\frac{dV}{dt} - \frac{4\pi L}{K}\frac{d^2P}{dt^2}.$$

At a maximum or minimum, $dV/dt = 0$. However, since $d^2V/dt^2 = -\left(\frac{4\pi L}{K}\right)\left(\frac{d^2P}{dt^2}\right)$ at $dV/dt = 0$, we see that, if d^2P/dt^2 is positive, only maxima can occur. Since more than one maximum necessitates minima between two maxima, there can be only one maximum if d^2P/dt^2 is positive.

Again, dP/dt is zero at $t = \infty$. Hence, if V were finite at $t = \infty$, dV/dt would be finite there. However, this is impossible. For, suppose V were finite and positive at $t = \infty$, then dV/dt would have been finite and negative for an infinite time, which is inconsistent with V being positive at $t = \infty$. If V were negative at $t = \infty$, dV/dt would be positive there, so that V would have passed through a minimum, which is inconsistent with what has already been established.

As regards the necessity of a positive value for d^2P/dt^2, it is clear that if P were of the exponential form, $P = P_0e^{-\alpha t}$, d^2P/dt^2 would be α^2P and would be positive. Again, if P were of the form of the sum of a number of exponentials

$$P = \sum P_m e^{-\alpha_m t}$$

with all the P_m's positive, d^2P/dt^2 would be $\sum \alpha_m^2 P_m e^{-\alpha_m t}$ and would be positive.

PROBLEM 2. *Proof that the curve V Plotted against t may show a point of inflection in the case of Fig. 4D.*

From (12)

$$\frac{d^2V}{dt^2} = -\frac{1}{RC}\frac{dV}{dt} - \frac{4\pi L}{K}\frac{d^2P}{dt^2} \tag{27}$$

and again, from (12)

$$\frac{d^2V}{dt^2} = \frac{1}{RC}\left(\frac{V}{RC} + \frac{4\pi L}{K}\frac{dP}{dt}\right) - \frac{4\pi L}{K}\frac{d^2P}{dt^2}$$

$$= \frac{1}{RC^2}\left[V + \frac{4\pi LRC}{K}\frac{dP}{dt} - \frac{4\pi L(RC)^2}{K}\frac{d^2P}{dt^2}\right].$$

If a point of inflection exists, it must be possible to determine t such that

$$V + \frac{4\pi L}{K}\left[RC\frac{dP}{dt} - (RC)^2\frac{d^2P}{dt^2}\right] = 0. \tag{28}$$

Let us consider the special case where $P = P_0e^{-\alpha t}$. Then the solution for V is given by (15), and (28) becomes

$$\frac{4\pi LP_0}{K}\left[\frac{RC\alpha}{1 - RC\alpha}(e^{-\alpha t} - e^{-t/RC}) - RC\alpha e^{-\alpha t} - (RC\alpha)^2e^{-\alpha t}\right] + V_0e^{-t/RC} = 0$$

$$\frac{4\pi LP_0}{K}\left[\frac{RC\alpha}{1 - RC\alpha} - RC\alpha - (RC\alpha)^2\right]e^{-\alpha t} + \left[V_0 - \frac{4\pi LP_0RC\alpha}{K(1 - RC\alpha)}\right]e^{-t/RC} = 0$$

$$\frac{4\pi LP_0}{K}\left[\frac{(RC\alpha)^3}{1 - RC\alpha}\right]e^{-\alpha t} + \left[V_0 - \frac{4\pi LP_0RC\alpha}{K(1 - RC\alpha)}\right]e^{-t/RC} = 0. \tag{29}$$

Now $\bar{V}_0$ is defined, in the present case, as

$$\bar{V}_0 \equiv 4\pi LP_0RC\alpha/K.$$

Hence (29) requires that

$$(RC\alpha)^2 + \left[\frac{V_0}{\bar{V}_0}(1 - RC\alpha) - 1\right] e^{-(1-RC\alpha)t/RC} = 0$$

$$-\frac{(1 - RC\alpha)t}{RC} = \log_e \frac{(RC\alpha)^2}{1 - V_0(1 - RC\alpha)/\bar{V}_0}.$$

The condition that t shall be real requires

$$1 - \frac{V_0}{\bar{V}_0} + RC\alpha\frac{V_0}{\bar{V}_0} > 0$$

$$1 - \frac{\bar{V}_0}{V_0} < RC\alpha.$$

Our conditions are that $\bar{V}_0 < V_0$, so that, since $RC\alpha$ is, in general, very small, a point of inflection is possible if, and only if, V_0 exceeds $\bar{V}_0$ by a very small amount.

SOME COORDINATE SYSTEMS ASSOCIATED WITH ELLIPTIC FUNCTIONS

BY

PARRY MOON [1] AND DOMINA EBERLE SPENCER [2]

The use of complex-plane transformations in potential theory is well-known and has resulted in the solution of a number of important two-dimensional problems. Among the analytic functions employed in these transformations are the Jacobian elliptic functions. The first application of elliptic-function transformations to potential theory seems to have been made by Siebeck (1) [3] in 1860. J. J. Thomson (2) in his book, *Recent Researches in Electricity and Magnetism* (1893), applied elliptic functions to the calculation of capacitance between various arrangements of metal plates. Further work on the subject has been done by Morton, Meyer, Michell, Petersohn, Holtzmüller, and others (3).

In all these cases, the purpose has been to solve a two-dimensional electrostatic or flow problem. The equipotentials and flow lines, if obtained, were regarded merely as specifying a two-dimensional field map and not as designating a coordinate system. Wangerin (4) (1877) and Bôcher (5) (1891) extended the work to three-dimensional coordinate systems based on elliptic functions. In none of the published results, however, are the new coordinate systems sufficiently developed so that they can be employed with facility by physicists and engineers.

The present paper gives the results of a study of cylindrical and rotational coordinate systems based on elliptic-function transformations. The six transformations listed in the paper are by no means all of the possible transformations utilizing elliptic functions. Other cases were considered, but these six appeared to be the ones most likely to be of practical utility.

1. THE TRANSFORMATIONS

A transformation,

$$z = \mathfrak{F}(w) \tag{1}$$

converts a complex quantity, $w = u + iv$, into another quantity, $z = x + iy$. The function $\mathfrak{F}$ is analytic and changes a rectangular grid, $u = \text{const.}$, $v = \text{const.}$, into another orthogonal map.

The following functions are employed: (1) the Jacobian sn-function, (2) the cn-function, (3) an inversion of (1), (4) ln sn w, (5) ln cn w,

[1] Department of Electrical Engineering, Massachusetts Institute of Technology, Cambridge, Mass.

[2] Department of Mathematics, University of Connecticut, Storrs, Conn.

[3] The boldface numbers in parentheses refer to the references appended to this paper.

TABLE I.—*Transformations.*

No.	Transformation	Fig. No.	Inversion of	x	y	ξ_{11}
	ELLIPTIC FUNCTIONS					
J1	$\xi = c\,\mathrm{sn}w$	1	J3	$\frac{c}{\Delta}\,\mathrm{sn}u\,\mathrm{dn}v$ where $\Delta = 1 - \mathrm{dn}^2u\,\mathrm{sn}^2v$.	$\frac{c}{\Delta}\,\mathrm{cn}u\,\mathrm{dn}u\,\mathrm{sn}v\,\mathrm{cn}v$	$\frac{c^2}{\Delta^2}(1 - \mathrm{sn}^2u\,\mathrm{dn}^2v)(\mathrm{dn}^2v - k^2\,\mathrm{sn}^2u)$
J2	$\bar{\xi} = c\,\mathrm{cn}w$	2	J2	$\frac{c}{\Delta}\,\mathrm{cn}u\,\mathrm{cn}v$	$\frac{c}{\Delta}\,\mathrm{sn}u\,\mathrm{dn}u\,\mathrm{sn}v\,\mathrm{dn}v$	$\frac{c^2}{\Delta^2}(\mathrm{sn}^2v + \mathrm{sn}^2u\,\mathrm{cn}^2v)(\mathrm{dn}^2v - k^2\,\mathrm{sn}^2u)$
J3	$\bar{\xi} = \frac{1}{c\,\mathrm{sn}w + ia} + \frac{i}{2a}$	3	J1	$\frac{AA}{A^2 + (B + aA)^2}$ where $A = c\,\mathrm{sn}u\,\mathrm{dn}v$,	$\frac{A(B + a\Delta)}{A^2 + (B + a\Delta)^2} - \frac{1}{2a}$ $B = c\,\mathrm{cn}u\,\mathrm{dn}u\,\mathrm{sn}v\,\mathrm{cn}v$.	$\frac{c^2A^2(1 - \mathrm{sn}^2u\,\mathrm{dn}^2v)(\mathrm{dn}^2v - k^2\,\mathrm{sn}^2u)}{[A^2 + (B + a\Delta)^2]^2}$
J4	$\bar{\xi} = \frac{c}{\pi}[\ln(1/k) - 2\ln\mathrm{sn}w]$	4	—	$\frac{c}{\pi}[\ln(1/k) + 2\ln\Delta - \ln(\mathrm{sn}^2u\,\mathrm{dn}^2v + \mathrm{cn}^2u\,\mathrm{dn}^2u\,\mathrm{sn}^2v\,\mathrm{cn}^2v)]$	$\frac{2c}{\pi}\tan^{-1}\left(\frac{\mathrm{cn}u\,\mathrm{dn}u\,\mathrm{sn}v\,\mathrm{cn}v}{\mathrm{sn}u\,\mathrm{dn}v}\right)$	$\left(\frac{2c}{\pi T}\right)^2[\mathrm{sn}^2u\,\mathrm{cn}^2u\,\mathrm{dn}^2u(\mathrm{cn}^2v - \mathrm{sn}^2v\,\mathrm{dn}^2v)^2 + \mathrm{sn}^2v\,\mathrm{cn}^2v\,\mathrm{dn}^2v(1 - k^2\,\mathrm{sn}^4u)^2]$, where $T = \mathrm{sn}^2u\,\mathrm{dn}^2v + \mathrm{cn}^2u\,\mathrm{dn}^2u\,\mathrm{sn}^2v\,\mathrm{cn}^2v$.
J5	$\bar{\xi} = \frac{2c}{\pi}\ln\mathrm{cn}w$	5	—	$\frac{c}{\pi}[\ln(\mathrm{cn}^2u\,\mathrm{cn}^2v + \mathrm{sn}^2u\,\mathrm{dn}^2u\,\mathrm{sn}^2v\,\mathrm{dn}^2v) - 2\ln\Delta]$	$\frac{2c}{\pi}\tan^{-1}\left(\frac{\mathrm{sn}u\,\mathrm{dn}u\,\mathrm{sn}v\,\mathrm{dn}v}{\mathrm{cn}u\,\mathrm{cn}v}\right)$	$\left(\frac{2c}{\pi Z}\right)^2[\mathrm{sn}^2u\,\mathrm{cn}^2u\,\mathrm{dn}^2u(\mathrm{dn}^2v - k'^2\,\mathrm{sn}^2v\,\mathrm{cn}^2v)^2 + \mathrm{sn}^2v\,\mathrm{cn}^2v\,\mathrm{dn}^2v(\mathrm{dn}^2u - k^2\,\mathrm{sn}^2u\,\mathrm{cn}^2u)^2]$, where $Z = \mathrm{cn}^2u\,\mathrm{cn}^2v + \mathrm{sn}^2u\,\mathrm{dn}^2u\,\mathrm{sn}^2v\,\mathrm{dn}^2v$.
J6	$\bar{\xi} = \frac{2Kc}{\pi}Z(w + iK') + ic$	6	—	$\frac{2Kc}{\pi}\left\{Z(u) + \frac{k^2}{\Delta'}\,\mathrm{sn}u\,\mathrm{cn}u\,\mathrm{dn}u\,\mathrm{sn}^2(v + K')\right\}$	$\frac{2Kc}{\pi}\left\{Z(v + K') + \frac{\pi v}{2KK'} - \frac{1}{\Delta'}\mathrm{dn}^2u \times \mathrm{sn}(v + K')\,\mathrm{cn}(v + K')\,\mathrm{dn}(v + K')\right\}$	$\left(\frac{2Kc}{\pi}\right)^2\left\{\frac{4k^4}{\Delta'^4}\mathrm{sn}^2u\,\mathrm{cn}^2u\,\mathrm{dn}^2u\,\mathrm{sn}^2(v + K') \times \mathrm{cn}^2(v + K')\,\mathrm{dn}^2(v + K') + \left[\mathrm{dn}^2u - \frac{E}{K} + \frac{k^2\,\mathrm{sn}^2(v + K')}{\Delta'^2}\left((\mathrm{dn}^2u\,\mathrm{cn}^2(v + K') \cdot(\mathrm{cn}^2u - \mathrm{sn}^2u\,\mathrm{dn}^2u) - k^2\,\mathrm{sn}^2u\cdot(1 - k^2\,\mathrm{sn}^4u)\right)\right]^2\right\}$, where $\Delta' = 1 - \mathrm{dn}^2u\,\mathrm{sn}^2(v + K')$.

NOTE. With all elliptic functions, the modulus k is associated with the variable u while k' is associated with v. For example, $\mathrm{sn}v = \mathrm{sn}(v, k')$, $\mathrm{dn}u = \mathrm{dn}(u, k)$.

and (6) the Jacobian zeta-function $Z(w)$. Details are listed in Table I, the metric coefficients being obtained from the relation,

$$g_{11} = g_{22} = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2. \tag{2}$$

The equations for x and y, given in Table I, allowed the calculation (6) of curves for u = const. and v = const. The resulting maps are shown in Figs. 1 to 6.

The value of an elliptic function depends of course on the modulus k as well as on the argument u or v. Figures 1 to 6 are calculated for $k^2 = 0.50$. By varying the modulus, we can obtain a whole series of maps for each elliptic function.

2. COORDINATE SYSTEMS

Separation of the real and imaginary parts of Eq. 1 yields two equations,

$$\begin{cases} x = \xi_1(u, v), \\ y = \xi_2(u, v), \end{cases} \tag{3}$$

which are given in Table I. A *cylindrical* coordinate system in 3-space

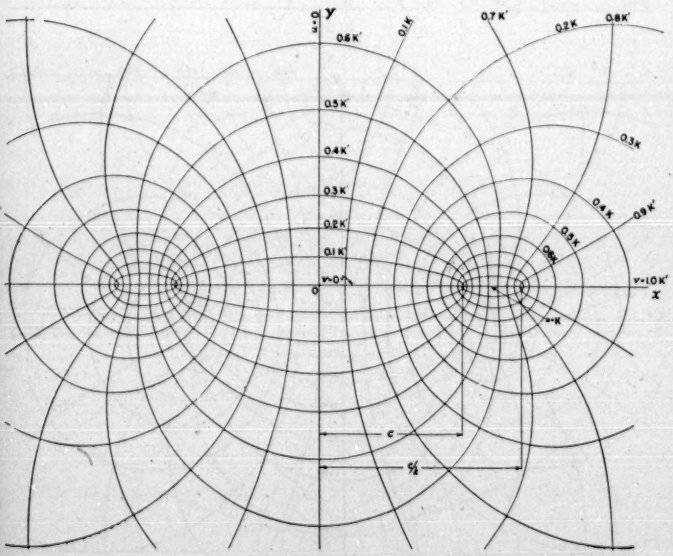


FIG. 1.

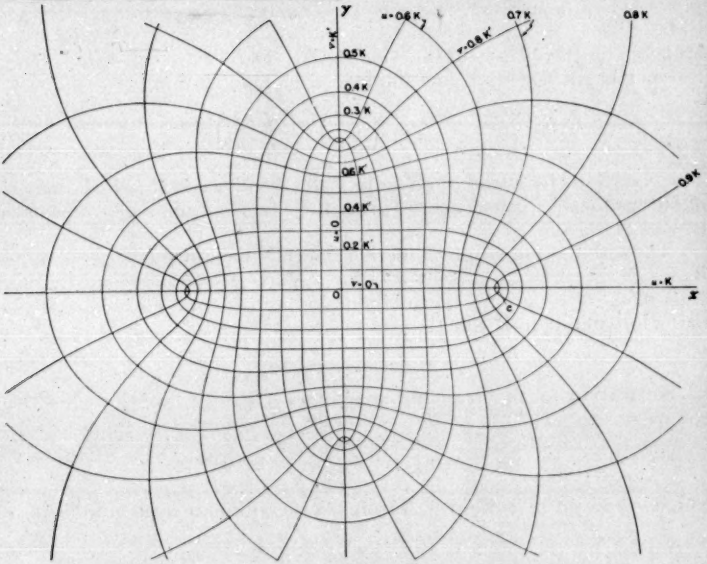

y
u=0.6K
0.7K
0.8K
v=0.8K'
v=K'
0.5K
0.4K
0.3K
0.6K'
0.4K'
0.2K'
u=0
v=0
0
u=K
x

FIG. 2.

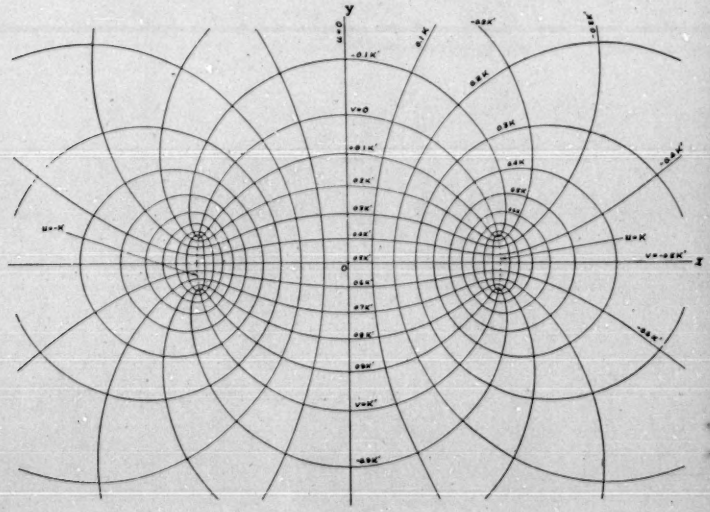

y
u=0
-0.1K'
0.1K
-0.2K'
0.2K
v=0
+0.1K'
0.3K
0.4K
-0.4K'
u=-K
u=K
v=-0.5K'
0
x

FIG. 3.

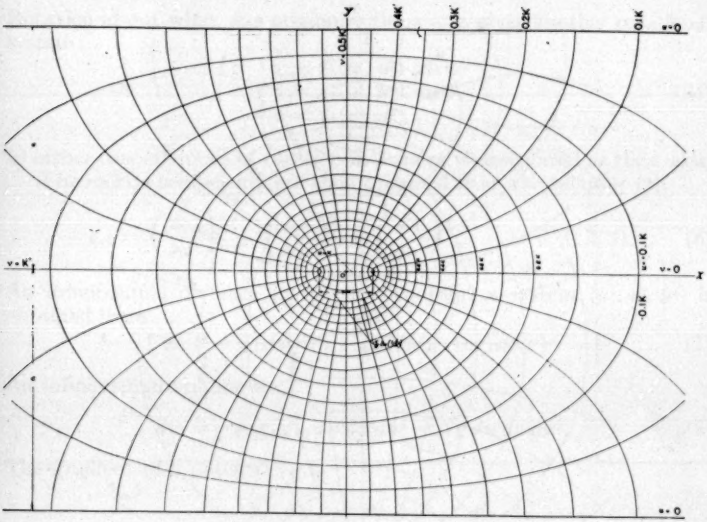

y
0.4K'
0.3K'
0.2K'
0.1K'
u=0
v=0.5K'
v=K'
u=0.1K
v=0
x
-0.1K
u=0

FIG. 4.

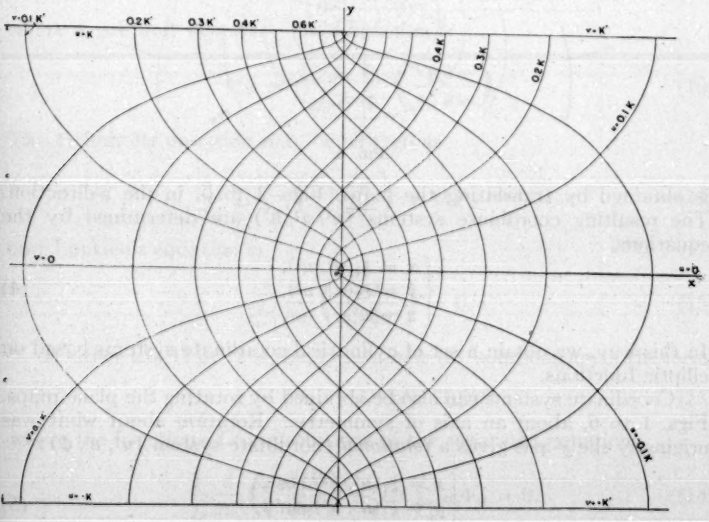

v=0.1K'
0.2K'
0.3K'
0.4K'
0.6K'
y
u=K
v=K'
0.4K
0.3K
0.2K
u=0.1K
v=0
u=0
x
v=-0.1K'
u=-K
v=-K'
u=-0.1K

FIG. 5.

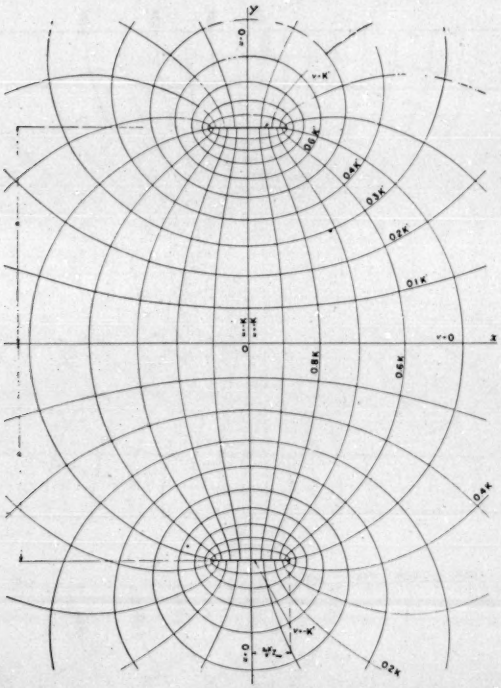


FIG. 6.

is obtained by translating the maps, Figs. 1 to 6, in the z-direction. The resulting coordinate systems (u^1, u^2, u^3) are determined by the equations,

$$\begin{cases} x = \xi_1(u^1, u^2), \\ y = \xi_2(u^1, u^2), \\ z = u^3. \end{cases} \tag{4}$$

In this way, we obtain a set of cylindrical coordinate systems based on elliptic functions.

Coordinate systems can also be obtained by rotating the plane maps, Figs. 1 to 6, about an axis of symmetry. Rotation about what was originally the y-axis gives a *rotational* coordinate system (u^1, u^2, ψ):

$$\begin{cases} x = \xi_1(u^1, u^2) \cos \psi, \\ y = \xi_1(u^1, u^2) \sin \psi, \\ z = \xi_2(u^1, u^2). \end{cases} \tag{5}$$

Rotation about what was originally the x-axis gives another rotational system:

$$\begin{cases} x = \xi_2(u^1, u^2)\cos\psi, \\ y = \xi_2(u^1, u^2)\sin\psi, \\ z = \xi_1(u^1, u^2). \end{cases} \tag{5a}$$

In either case, the axis of rotation in 3-space is designated as the z-axis.

The metric coefficients are then obtained from the relation (7),

$$g_{ii} = \left(\frac{\partial x}{\partial u^i}\right)^2 + \left(\frac{\partial y}{\partial u^i}\right)^2 + \left(\frac{\partial z}{\partial u^i}\right)^2, \qquad (i = 1, 2, 3). \tag{6}$$

An infinitesimal distance in the new coordinate system (u^1, u^2, u^3) is obtained from

$$(ds)^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 + g_{33}(du^3)^2. \tag{7}$$

An infinitesimal volume is

$$d\mathcal{V} = \sqrt{g_{11}g_{22}g_{33}}\, du^1du^2du^3 = \sqrt{g}\, du^1du^2du^3. \tag{8}$$

The gradient of a potential φ is

$$\nabla\varphi = \frac{\mathbf{a}_1}{\sqrt{g_{11}}}\frac{\partial\varphi}{\partial u^1} + \frac{\mathbf{a}_2}{\sqrt{g_{22}}}\frac{\partial\varphi}{\partial u^2} + \frac{\mathbf{a}_3}{\sqrt{g_{33}}}\frac{\partial\varphi}{\partial u^3}, \tag{9}$$

where $\mathbf{a}_i$ are unit vectors. The Laplacian is

$$\nabla^2\varphi = \frac{1}{\sqrt{g}}\sum_{i=1}^{3}\frac{\partial}{\partial u^i}\left(\frac{\sqrt{g}}{g_{ii}}\frac{\partial\varphi}{\partial u^i}\right). \tag{10}$$

The Helmholtz equation may be written as

$$\frac{1}{\sqrt{g}}\sum_{i=1}^{3}\frac{\partial}{\partial u^i}\left(\frac{\sqrt{g}}{g_{ii}}\frac{\partial\varphi}{\partial u^i}\right) + (k)^2\varphi = 0 \tag{11}$$

and Laplace's equation is

$$\sum_{i=1}^{3}\frac{\partial}{\partial u^i}\left(\frac{\sqrt{g}}{g_{ii}}\frac{\partial\varphi}{\partial u^i}\right) = 0. \tag{12}$$

If either Eq. 11 or 12 is *simply separable* (8), solutions are of the form,

$$\varphi = U^1(u^1)\cdot U^2(u^2)\cdot U^3(u^3), \tag{13}$$

where the functions U^i are solutions of the separated equations,

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\alpha_j\Phi_{ij} = 0. \tag{14}$$

TABLE II.—*Cylindrical Coordinate Systems.*

No.	Transformation	Fig. No.	Equations	Metric Coefficients	Simple Separability				One-dim. Solution		α_{11}
					Helm.		Lap.				
					$\phi(u^1, u^2, u^3)$	$\phi(u^1, u^2)$	$\phi(u^1, u^2, u^3)$	$\phi(u^1, u^2)$	Helm.	Lap.	
J1C	$z = c\,\mathrm{snw}$	1	$x = \frac{c}{\Lambda}\,\mathrm{sn}\mu\,\mathrm{dn}\nu,$ $y = \frac{c}{\Lambda}\,\mathrm{cn}\mu\,\mathrm{dn}\mu\,\mathrm{sn}\nu\,\mathrm{cn}\nu,$ $z = z,$ where $\Lambda = 1 - \mathrm{dn}^2\mu\,\mathrm{sn}^2\nu.$	$g_{11} = g_{22} =$ $\frac{c^2}{\Lambda^2}(1 - \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu)(\mathrm{dn}^2\nu - k^2\,\mathrm{sn}^2\mu),$ $g_{33} = 1.$	×	×	×	●	×	●	
J2C	$\bar{z} = c\,\mathrm{cnw}$	2	$x = \frac{c}{\Lambda}\,\mathrm{cn}\mu\,\mathrm{cn}\nu,$ $y = \frac{c}{\Lambda}\,\mathrm{sn}\mu\,\mathrm{dn}\mu\,\mathrm{sn}\nu\,\mathrm{dn}\nu,$ $z = z,$ where $\Lambda = 1 - \mathrm{dn}^2\mu\,\mathrm{sn}^2\nu.$	$g_{11} = g_{22} =$ $\frac{c^2}{\Lambda^2}(\mathrm{sn}^2\nu + \mathrm{sn}^2\mu\,\mathrm{cn}^2\nu)(\mathrm{dn}^2\nu - k^2\,\mathrm{sn}^2\mu),$ $g_{33} = 1.$	×	×	×	●	×	●	
J3C	$\bar{z} = \frac{1}{c\,\mathrm{snw} + ia} + \frac{i}{2a},$ $a = c/\sqrt{k}.$	3	$x = \frac{\Lambda\,\mathrm{sn}\mu\,\mathrm{dn}\nu}{c[\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + \Theta^2]},$ $y = \frac{\Lambda\Theta}{c[\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + \Theta^2]} - \frac{\sqrt{k}}{2c},$ $z = z,$ where $\Lambda = 1 - \mathrm{dn}^2\mu\,\mathrm{sn}^2\nu,$ $\Theta = \mathrm{cn}\mu\,\mathrm{dn}\mu\,\mathrm{sn}\nu\,\mathrm{cn}\nu + \Lambda/\sqrt{k}.$	$g_{11} = g_{22} = \frac{\Lambda^2\alpha^2\lambda^2}{[\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + \Theta^2]^2},$ $g_{33} = 1,$ where $\alpha^2 = \mathrm{dn}^2\nu - k^2\,\mathrm{sn}^2\mu,$ $\lambda^2 = 1 - \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu.$	×	×	×	●	×	●	
J4C	$\bar{z} = \frac{c}{\pi}[\ln(1/k) - 2\ln\mathrm{snw}]$	4	$x = \frac{c}{\pi}[\ln(1/k) + 2\ln\Lambda - \ln(\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + \mathrm{cn}^2\mu\,\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu)],$ $y = \frac{2c}{\pi}\tan^{-1}\left(\frac{\mathrm{cn}\mu\,\mathrm{dn}\mu\,\mathrm{sn}\nu\,\mathrm{cn}\nu}{\mathrm{sn}\mu\,\mathrm{dn}\nu}\right),$ $z = z,$ where $\Lambda = 1 - \mathrm{dn}^2\mu\,\mathrm{sn}^2\nu.$	$g_{11} = g_{22} =$ $\left(\frac{2c}{\pi\Upsilon}\right)^2[\mathrm{sn}^2\mu\,\mathrm{cn}^2\mu\,\mathrm{dn}^2\mu$ $\times(\mathrm{cn}^2\nu - \mathrm{sn}^2\nu\,\mathrm{dn}^2\nu)^2$ $+ \mathrm{sn}^2\nu\,\mathrm{cn}^2\nu\,\mathrm{dn}^2\nu(1 - k^2\,\mathrm{sn}^4\mu)^2],$ $g_{33} = 1,$ where $\Upsilon = \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + \mathrm{cn}^2\mu\,\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu.$	×	×	×	●	×	●	

TABLE II.—*Continued.*

No.	Transformation	Fig. No.	Equations	Metric Coefficients	Simple Separability				One-dim. Solution		α_{11}
					Helm.		Lap.				
					$\phi(u^1, u^2, u^3)$	$\phi(u^1, u^2)$	$\phi(u^1, u^2, u^3)$	$\phi(u^1, u^2)$	Helm.	Lap.	
J5C	$\Xi = \frac{2c}{\pi} \ln \operatorname{cn} w$	5	$x = \frac{c}{\pi}\left[\ln(\operatorname{cn}^2\mu \operatorname{cn}^2\nu + \operatorname{sn}^2\mu \operatorname{dn}^2\mu \operatorname{sn}^2\nu \operatorname{dn}^2\nu) - 2 \ln A\right]$, $y = \frac{2c}{\pi} \tan^{-1}\left(\frac{\operatorname{sn}\mu \operatorname{dn}\mu \operatorname{sn}\nu \operatorname{dn}\nu}{\operatorname{cn}\mu \operatorname{cn}\nu}\right)$, $z = z$.	$g_{11} = g_{22} = \left(\frac{2c}{\pi\Xi}\right)^2 [\operatorname{sn}^2\mu \operatorname{cn}^2\mu \operatorname{dn}^2\mu \times (\operatorname{dn}^2\nu - k'^2 \operatorname{sn}^2\nu \operatorname{cn}^2\nu)^2 + \operatorname{sn}^2\nu \operatorname{cn}^2\nu \operatorname{dn}^2\nu \times (\operatorname{dn}^2\mu - k^2 \operatorname{sn}^2\mu \operatorname{cn}^2\mu)^2]$, $g_{33} = 1$, where $\Xi = \operatorname{cn}^2\mu \operatorname{cn}^2\nu + \operatorname{sn}^2\mu \operatorname{dn}^2\mu \operatorname{sn}^2\nu \operatorname{dn}^2\nu$.	×	×	×	●	×	●	
J6C	$\Xi = \frac{2Kc}{\pi} Z(w + iK') + ic$.	6	$x = \frac{2Kc}{\pi}\left\{Z(\mu) + \frac{k^2}{\Delta'} \operatorname{sn}\mu \operatorname{cn}\mu \operatorname{dn}\mu \operatorname{sn}^2(\nu + K')\right\}$, $y = \frac{2Kc}{\pi}\left\{Z(\nu + K') + \frac{\pi\nu}{2KK'} - \frac{1}{\Delta'} \operatorname{dn}^2\mu \operatorname{sn}(\nu + K') \operatorname{cn}(\nu + K') \operatorname{dn}(\nu + K')\right\}$, $z = z$, where $\Delta' = 1 - \operatorname{dn}^2\mu \operatorname{sn}^2(\nu + K')$.	$g_{11} = g_{22} = \left(\frac{2Kc}{\pi}\right)^2\left\{\frac{4k^4}{\Delta'^4} \operatorname{sn}^2\mu \operatorname{cn}^2\mu \operatorname{dn}^2\mu \operatorname{sn}^2(\nu + K') \operatorname{cn}^2(\nu + K') \operatorname{dn}^2(\nu + K') + \left[\operatorname{dn}^2\mu - \frac{E}{K} + \frac{k^2 \operatorname{sn}^2(\nu + K')}{\Delta'^2}\left(\operatorname{dn}^2\mu \operatorname{cn}^2(\nu + K') \cdot (\operatorname{cn}^2\mu - \operatorname{sn}^2\mu \operatorname{dn}^2\mu) - k^2 \operatorname{sn}^2\mu(1 - k^2 \operatorname{sn}^4\mu)\right)\right]^2\right\}$, $g_{33} = 1$.	×	×	×	●	×	●	

NOTE. Crosses indicate non-separability, dots indicate separability. The modulus k is associated with the variable μ, and k' is associated with ν: e.g., $\operatorname{sn}\mu = \operatorname{sn}(\mu, k)$, $\operatorname{sn}\nu = \operatorname{sn}(\nu, k')$.

Here Φ_{ij} are elements (8) of the Stäckel matrix $[S]$, α_j are separation constants, and f_i are obtained from

$$\frac{\sqrt{g}}{S} = f_1(u^1) \cdot f_2(u^2) \cdot f_3(u^3). \tag{15}$$

If the partial differential Eq. 12 is not simply separable, there is still a possibility of obtaining solutions that contain an arbitrary function $R(u^1, u^2, u^3)$:

$$\varphi = \frac{U^1(u^1) \cdot U^2(u^2) \cdot U^3(u^3)}{R(u^1, u^2, u^3)}. \tag{16}$$

Separability conditions are discussed elsewhere (8). With *R-separability*, the functions U^i are obtained by solving the ordinary differential equation (14), just as in simple separability.

3. CYLINDRICAL COORDINATE SYSTEMS

From the six transformations of Table I are obtained the six cylindrical coordinate systems listed in Table II. To indicate that these are 3-dimensional systems, we have replaced u and v by μ and ν, the three coordinates being

$$u^1 = \mu, \qquad u^2 = \nu, \qquad u^3 = z.$$

The equations for the coordinate surface are obtained from Eq. 4. In all cases, the metric coefficients g_{11} and g_{22} are equal and are the same as in Table I, while g_{33} is always unity.

R-separation is never possible in a cylindrical coordinate system (8). Criteria for simple separability have been applied to the systems of Table II, and the results are included in the table. Note that the Helmholtz equation does not separate in any of these coordinates and the Laplace equation separates only if the potential is independent of z. A one-dimensional solution,

$$\varphi = \varphi(\mu) \qquad \text{or} \qquad \varphi = \varphi(\nu),$$

is possible with Laplace's equation but not with the Helmholtz equation.

In all cylindrical coordinate systems, when φ is independent of z, the separation equations are

$$\begin{cases} \dfrac{d^2M}{d\mu^2} - p^2M = 0, \\ \dfrac{d^2N}{d\nu^2} + p^2N = 0, \end{cases}$$

and the solutions are

$$\varphi = M(\mu) \cdot N(\nu) = \begin{cases} \sinh p\mu \sin p\nu \\ \cosh p\mu \cos p\nu \end{cases}$$

TABLE III. Rotational Coordinate Systems.

No.	Transformation	Fig. No.	Equations	Metric Coefficients	Separability: Helm. $\phi(u^1, u^2, u^3)$	Separability: Helm. $\phi(u^1, u^2)$	Separability: Lap. $\phi(u^1, u^2, u^3)$	Separability: Lap. $\phi(u^1, u^3)$	One-dim. Solution: Helm.	One-dim. Solution: Lap.	α_1
J1Ry	$z = c\,\mathrm{sn}w$	1 Rot. about y axis.	$x = \frac{c}{\Lambda}\,\mathrm{sn}\mu\,\mathrm{dn}\nu\cos\psi$, $y = \frac{c}{\Lambda}\,\mathrm{sn}\mu\,\mathrm{dn}\nu\sin\psi$, $z = \frac{c}{\Lambda}\,\mathrm{cn}\mu\,\mathrm{dn}\mu\,\mathrm{sn}\nu\,\mathrm{cn}\nu$, where $\Lambda = 1 - \mathrm{dn}^2\mu\,\mathrm{sn}^2\nu$.	$g_{11} = g_{22} = \frac{c^2}{\Lambda^2}(1 - \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu)(\mathrm{dn}^2\nu - k^2\,\mathrm{sn}^2\mu)$, $g_{33} = \frac{c^2}{\Lambda^2}\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu$.	×	×	R	R	×	×	−1
J1Rx	$z = c\,\mathrm{sn}w$	1 Rot. about x axis.	$x = \frac{c}{\Lambda}\,\mathrm{cn}\mu\,\mathrm{dn}\mu\,\mathrm{sn}\nu\,\mathrm{cn}\nu\cos\psi$, $y = \frac{c}{\Lambda}\,\mathrm{cn}\mu\,\mathrm{dn}\mu\,\mathrm{sn}\nu\,\mathrm{cn}\nu\sin\psi$, $z = \frac{c}{\Lambda}\,\mathrm{sn}\mu\,\mathrm{dn}\nu$.	$g_{11} = g_{22} = \frac{c^2}{\Lambda^2}(1 - \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu)(\mathrm{dn}^2\nu - k^2\,\mathrm{sn}^2\mu)$, $g_{33} = \frac{c^2}{\Lambda^2}\mathrm{cn}^2\mu\,\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu$.	×	×	R	R	×	×	−1
J2R	$\bar{z} = c\,\mathrm{cn}w$	2 Rot. about y axis.	$x = \frac{c}{\Lambda}\,\mathrm{cn}\mu\,\mathrm{cn}\nu\cos\psi$, $y = \frac{c}{\Lambda}\,\mathrm{cn}\mu\,\mathrm{cn}\nu\sin\psi$, $z = \frac{c}{\Lambda}\,\mathrm{sn}\mu\,\mathrm{dn}\mu\,\mathrm{sn}\nu\,\mathrm{dn}\nu$, where $\Lambda = 1 - \mathrm{dn}^2\mu\,\mathrm{sn}^2\nu$.	$g_{11} = g_{22} = \frac{c^2}{\Lambda^2}(\mathrm{sn}^2\nu + \mathrm{sn}^2\mu\,\mathrm{cn}^2\nu)(\mathrm{dn}^2\nu - k^2\,\mathrm{sn}^2\mu)$, $g_{33} = \frac{c^2}{\Lambda^2}\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu$.	×	×	R	R	×	×	−1
J3R	$\bar{z} = \frac{1}{c\,\mathrm{sn}w + ia} + \frac{i}{2a}$	3 Rot. about y axis.	$x = \frac{\Lambda\,\mathrm{sn}\mu\,\mathrm{dn}\nu}{c[\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + \Theta^2]}\cos\psi$, $y = \frac{\Lambda\,\mathrm{sn}\mu\,\mathrm{dn}\nu}{c[\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + \Theta^2]}\sin\psi$, $z = \frac{\Lambda\Theta}{c[\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + \Theta^2]} - \frac{\sqrt{k}}{2c}$, where $\Lambda = 1 - \mathrm{dn}^2\mu\,\mathrm{sn}^2\nu$, $\Theta = \mathrm{cn}\mu\,\mathrm{dn}\mu\,\mathrm{sn}\nu\,\mathrm{cn}\nu + \Lambda/\sqrt{k}$.	$g_{11} = g_{22} = \frac{\Lambda^2\alpha^2\lambda^2}{[\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + \Theta^2]^2}$, $g_{33} = \frac{\Lambda^2\,\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}{[\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + \Theta^2]^2}$, where $\alpha^2 = \mathrm{dn}^2\nu - k^2\,\mathrm{sn}^2\mu$, $\lambda^2 = 1 - \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu$.	×	×	R	R	×	×	−1

NOTE. Crosses indicate non-separability, R indicates R-separability. The modulus k is associated with μ, k' is associated with ν: $\mathrm{sn}\mu = \mathrm{sn}(\mu, k)$ $\mathrm{sn}\nu = \mathrm{sn}(\nu, k')$.

or

$$\varphi = \begin{cases} \sinh q\nu \sin q\mu \\ \cosh q\nu \cos q\mu. \end{cases}$$

If solutions are one dimensional, the solution in any cylindrical system is

$$\varphi = A + B\mu \qquad \text{or} \qquad \varphi = A + B\nu.$$

Thus, elementary functions suffice even in the rather elaborate cylindrical systems defined in terms of elliptic functions.

We conclude that the six cylindrical coordinate systems may be used in the solution of Laplace's equation for all cases in which the potential is independent of z and is known on coordinate surfaces enclosing a portion of space. With sn-cylindrical coordinates (J1C), for instance, we can obtain the potential distribution produced by two long, charged, metal strips $\mu = \pm K$ (Fig. 1). Or long metal bars, having the oval cross-section given by $\mu = \text{const}$, may be used; or surfaces specified by $\nu = \text{const}$. Coordinates J3C will give the field between two portions of a long circular cylinder with any specified gap between the two parts. Zeta-function coordinates J6C may be employed to obtain the field between two parallel plates of a capacitor. Applications, however, are not limited to cases where the potential is constant on the boundary surfaces. Any known potential distribution on the boundaries or any known field strength at the boundaries will allow a solution.

4. ROTATIONAL COORDINATE SYSTEMS

From the six transformations of Table I can be obtained four rotational coordinate systems that are R-separable (Table III). The equations for the coordinate surfaces are derived from Eqs. 5 or 5a, and the metric coefficients are obtained from those given in Table I. Transformations J4, J5, and J6 yield rotational coordinate systems; but these systems do not allow any kind of separation, so they have not been included in Table III. Transformations J1, J2, and J3 give rotational coordinate systems which allow R-separation of the Laplace equation.

When the map of Fig. 1 is rotated about the y-axis, we obtain an orthogonal family of ring cyclides, rotation cyclides, and meridian planes. This system has been considered by Wangerin (4) and by Bôcher (5). One application is to the electrostatic field of a charged flat ring ($\mu = K$) or a ring of oval cross section ($\mu = \text{const.}$). The corresponding problem for rotation about the x-axis (J1Rx coordinates) deals with the field between two collinear wires ($\mu = K$) of finite length separated by distance $2c$.

Rotation of the map, Fig. 2, about either axis gives rotational system J2R (Table III). An application is the electric conduction field between two long wires ($\nu = K'$), where flow is restricted to a circular opening ($\nu = 0$) by a non-conducting ring $\mu = K$.

Rotation of Fig. 3 about either axis produces the coordinate system

J3R. An example of this system is the field between two spherical caps ($\nu = 0, \nu = K'$) with a gap between them. The relative size of cap and gap is altered by varying the modulus k. In considering such examples, one should realize that (unlike the cylindrical cases) the flux lines and equipotentials do not in general follow the contours of Figs. 1 to 3. The R-function in the solution changes the appearance of the entire map. Nevertheless, the rotational coordinate systems of Table III have a wide variety of practical applications in both Dirichlet and Neumann problems.

REFERENCES

(1) SIEBECK, "Über eine Gattung von Curven vierten Grades, welche mit den elliptischen Funktionen zusammenhängen," *J. reine u. ung. Math.*, Vol. 57, p. 359 (1860).

(2) J. J. THOMSON, "Recent Researches in Electricity and Magnetism", Oxford Univ. Press, 1893, pp. 236–250.

(3) W. B. MORTON, "On the Parallel-Plate Condenser, and Other Two-Dimensional Fields Specified by Elliptic Functions," *Phil. Mag.*, Vol. 2, p. 827 (1926);

E. MEYER, "Zwei Beispiele zweidimensionaler elektrostatischer Kraftlinienbilder," *Math. Ann.*, Vol. 93, p. 157 (1925);

J. H. MICHELL, "A Map of the Complex Z-Function: a Condenser Problem," *Mess. of Math.*, Vol. 23, p. 72 (1894);

H. PETERSOHN, "Zweidimensionale elektrostatische Probleme," *Zeits. f. Phys.*, Vol. 38, p. 727 (1926);

G. HOLTZMÜLLER, "Beiträge zur Theorie der isogonalen Verwandtschaften," *Zeits. f. Math. u. Phys.*, Vol. 18, p. 227 (1873); Vol. 20, p. 1 (1875); "Einführung in die Theorie der isogonalen Verwandtschaften," Leipzig, B. G. Teubner, 1882;

F. BOWMAN, "Notes on Two-Dimensional Field Problems," *London Math. Soc., Proc.*, Vol. 39, p. 205 (1935);

W. G. BICKLEY, "Two-Dimensional Potential Problems Concerning a Single Closed Boundary," *Roy. Soc. Phil. Trans.*, London, Vol. 228A, p. 235 (1928–9); "Two-Dimensional Potential Problems for the Space Outside a Rectangle," *London Math. Soc., Proc.*, Vol. 37, p. 82 (1934);

T. J. I'A BROMWICH, "Note on a Condenser Problem," *Mess. of Math.*, Vol. 31, p. 184 (1902);

A. E. H. LOVE, "Some Electrostatic Distributions in Two Dimensions," *London Math. Soc., Proc.*, Vol. 22, p. 337 (1924);

W. M. PAGE, "Some Two-Dimensional Problems in Electrostatics and Hydrodynamics," *London Math. Soc., Proc.*, Vol. 11, p. 313 (1913);

E. G. C. POOLE, "On the Discontinuous Motion Produced in an Infinite Stream by Two Plane Obstacles," *London Math. Soc., Proc.*, Vol. 22, p. 425 (1924);

F. OBERHETTINGER AND W. MAGNUS, "Zur Berechnung des Wellenwiderstandes einer Bandleitung mit kreisförmigem bzw. rechteckigem Aussenleiterquerschnitt," *Archiv. f. Elek.*, Vol. 37, p. 380 (1943); "Anwendung der elliptischen Funktionen in Physik und Technik," Berlin, Springer, 1949.

(4) A. WANGERIN, "Reduction der Potentialgleichung für gewisse Rotationskörper auf eine gewöhnliche Differentialgleichung," Preisschriften der Jablonowski'schen Gesell., Leipzig, No. 18, 1875; "Über ein dreifach orthogonales Flächensystem, gebildet aus gewissen Flächen vierter Ordnung," *J. f. reine und ungewandte Math.*, Vol. 82, pp. 145, 348 (1877).

(5) M. BÔCHER, "Über die Reihenentwickelungen der Potentialtheorie," Göttingen, 1891.

(6) G. W. AND R. M. SPENCELEY, "Smithsonian Elliptic Functions Tables," Washington, D. C., Smithsonian Institution, 1947.

(7) PARRY MOON AND D. E. SPENCER, "Cylindrical and Rotational Coordinate Systems," JOUR. FRANKLIN INST., Vol. 252, p. 327 (1951).

(8) PARRY MOON AND D. E. SPENCER, "Separability Conditions for the Laplace and Helmholtz Equations," JOUR. FRANKLIN INST., Vol. 253, p. 585 (1952).

Enzymatic System Isolated.—Enzyme chemists from the University of Wisconsin have announced the isolation of the complete enzymatic system by which living things obtain heat and energy from fatty acids, thereby cracking one of the most formidable barriers to an understanding of a key life process. Dr. David E. Green of the University of Wisconsin Enzyme Institute gave the details of the discovery during a symposium of fatty acid metabolism, at the Annual Meeting of the American Chemical Society, in Los Angeles.

The discovery will give scientists a new and more intimate chemical approach to some of the problems of cancer and diabetes, diseases which are characterized by defective links in the enzyme complex.

The announcement brings to a close an intense effort by enzyme scientists the world over to gain an insight into the process by which fatty acids—key fuel stuffs in the cells of animals—are burned to supply heat and energy for all life activities from muscular action to thinking. The fatty acid fuels are one of the two basic energy substances used by living things. The other is sugar. The enzyme system by which sugar is broken down and used for energy is already known, but scientists have been working for years on the equally important enzyme complex which oxidizes fatty acids.

The Wisconsin scientists reported they found that the fatty acid oxidation is carried out by a team of nine enzymes, all working together like the cogs of an expensive watch. The scientists isolated each enzyme in a state of high purity. A key enzyme in the whole process, and the one that proved the most difficult to isolate, is a deep green in color. This enzyme contains a vitamin known as flavin as an essential part; scientists did not hitherto know that this particular flavin existed.

Work with the fatty acid complex is rendered tremendously difficult by the fact that each of the intermediates in the process is extremely unstable, and experimental conditions must be rigidly controlled to keep the enzymes and the chemical products from disintegrating. The scientists virtually recreate a basic life process in a test tube.

Dr. Green pointed out that the discovery conclusively shows that modern scientific work is based on teamplay. Eight Wisconsin scientists worked on the job of unveiling the mystery of the enzyme system for oxidizing fatty acids.

New Magnetic Material.—A new "soft" magnetic material composed of noncritical metals has been developed by the Naval Ordnance Laboratory. Named 16-Alfenol, the new material is composed of 16% aluminum and 84% iron. The alloy itself is not new and was used by the Japanese during WW II with some success. Containing magnetic properties which should make it valuable in many applications, 16-Alfenol with properties superior to silicon iron will be especially useful in transformer cores such as are commonly used in high frequency communications instruments.

NOTES FROM THE NATIONAL BUREAU OF STANDARDS*

TRANSITIONS IN TEFLON

Because Teflon (polytetrafluoroethylene) is highly resistant to chemical and thermal degradation, this plastic is proving very useful in gaskets, washers, valve packings, and similar applications. However, where dimensional stability is required, consideration must be given to the transitions—always accompanied by sharp changes in volume—that occur between different polymorphic forms of this polymeric material with changes in temperature and pressure. In the course of a continuing study of the effect of pressure on high polymers, C. E. Weir of the National Bureau of Standards has found that Teflon is unique among all high polymers investigated to date in that it exists in three polymorphic forms within the range from 10° to 80° C. at pressures between one and 10,000 atmospheres. Two different transitions between phases may occur at room temperature, depending on the pressure to which the specimen is subjected.

The work on Teflon was carried out as part of a broad program of research now under way at NBS on the fundamental properties of high polymers. Polymeric substances—such as rubber, plastics, textiles, leather, and papers—owe their elasticity, plasticity, and other special qualities to the long, chainlike structure of their molecules. The Bureau is investigating the basic constants and properties of these materials in order to learn more about their molecular structure and thus to design new polymers having specified characteristics. A knowledge of the conditions under which transitions take place in high polymers is needed in order to predict the temperature range over which a given polymer will have useful mechanical properties.

In the NBS experiments, the Teflon specimen was immersed at different temperatures in a suitable pressure transmitting liquid within the smooth bore of a thick-walled bomb of hardened steel. Pressure was applied to the specimen by forcing a leak-proof piston into the bomb. At intervals the depth of penetration of the piston and the corresponding hydrostatic pressure were measured. When pressure was plotted against piston position (an indication of the extent of volume change) the transitions between polymorphic forms appeared as discontinuities in the curve.

The entire apparatus, including a hydraulic press for the application of pressure, was enclosed in a thermostatically controlled air bath. Temperatures were measured with a calibrated mercury thermometer

* Communicated by the Director.

which was set in a well in a massive brass block fastened to and in intimate thermal contact with the pressure vessel. Depth of penetration was determined to 0.0001 in. by means of a dial gage mounted on the ram used to drive the piston into the vessel. Internal hydrostatic pressure was measured with a manganin pressure gage, which makes use of the very small change in the resistance of a coil of manganin wire with pressure. To take into account effects due to compression of the confining liquid, the piston and ram, the packing washers, and other parts, as well as distortion in the bore of the bomb, a comparison experiment was performed in which the specimen was replaced by a steel bar of similar volume.

Teflon "melts" at 330° C., passing from a crystalline to an amorphous phase. Previous investigations at NBS and elsewhere had shown the existence of two other transition points at room temperature, one occurring under atmospheric pressure, the other under a pressure of 5500 atmospheres. The first of these transitions takes place at about 20° C. and involves approximately a 1-per cent change in volume. Actually, this transition, which is reversible and involves a change in the X-ray diffraction pattern, is not a simple effect but rather a complex transition appearing to consist of two overlapping volume discontinuities, one centered at 20° C. and the other at 30° C. The second room-temperature transition, at 5500 atmospheres, takes place at 25.5° C. and involves a volume change of about 2.5 per cent.

The NBS studies show that on a pressure-temperature chart, or phase diagram, the two room-temperature transitions are points on two different transition lines which, together with a third transition line, intersect in a triple point and form the boundaries between three polymorphic forms of Teflon, designated phases I, II, and III. The room-temperature transition at atmospheric pressure is the beginning of a transition line between phases I and II which proceeds to higher pressure at an accelerating rate with increasing temperature and finally terminates in the triple point at about 70° C. At the higher pressures the volume change associated with this transition line decreases rapidly to a very small value. The transition line between phases II and III leaves the triple point with negative slope, proceeding toward lower temperatures and higher pressures and ultimately passing through the high-pressure room-temperature transition point previously observed at 5500 atmospheres. The average negative latent heat involved in crossing this transition line has been calculated to be about 29 joules/gram.

The third transition line, between phases I and III, passes from the triple point to higher pressures with increasing temperatures. Within the region studied, this transition line is also accompanied by a volume change of about 2.5 per cent. However, because of experimental limi-

tations, this line could not be followed to temperatures higher than 81° C.

Although certain thermodynamic relationships do not appear to be rigidly fulfilled in the region of the experimental triple point, the discrepancies can probably be ascribed to experimental error and the fact that the transitions occur over finite intervals of pressure and temperature. Extremely sharp transitions would not be expected inasmuch as Teflon, in all probablility, does not consist of a single molecular species.

NOTE: For further details, see "Transitions and Phases of Teflon," by C. E. Weir, *J. Research NBS*, Vol. 50, p. 95 (1953), reprinted as NBS Research Paper RP 2395, 5 cents from Superintendent of Documents, U. S. Government Printing Office, Washington, D. C.

RESONANCE-TYPE VIBRATION CALIBRATOR

A resonance-type vibration calibrator recently constructed at the National Bureau of Standards combines unusual simplicity and economy of construction with pure waveform and high sensitivity. The calibrator consists essentially of a system of mechanically resonant beams excited by an ordinary 8-in. loudspeaker; the transducer being calibrated is fastened rigidly to the mechanical system, and the amplitude of vibration is observed optically with reference to a fixed point in space. Designed by Frank C. Smith of the Bureau's sound laboratory, the device has been used to calibrate moving-coil type velocity gages over a frequency range from 10 to 250 cycles per second, at displacement amplitudes of as little as 50 microinches.

Although most vibration pickups are calibrated by the manufacturer, it is often desirable to have available adequate calibration facilities at the laboratory using the pickup. A number of methods of calibrating transducers over various ranges of frequency and amplitude have been developed, but many of these methods require elaborate and expensive equipment. Furthermore, many calibrators are not satisfactory at the small displacement amplitudes at which some sensitive pickups are actually used. Although relatively simple in construction, the NBS equipment can be used to calibrate sensitive transducers at low levels of acceleration, of the order of 0.05 to 5 gravities.

The resonant beams of the NBS device consist of two parallel plates of high-strength aluminum alloy or medium-carbon steel, each about 18 in. long. The plates used at the Bureau are about $2\frac{1}{2}$ in. in width and range in thickness from $\frac{1}{16}$ to $\frac{1}{4}$ in. By using various thicknesses and materials, different stiffnesses and thus different resonant frequencies can be obtained. The plates are clamped rigidly to a steel frame at their lower ends and to a block at their free ends; this provides a mass-spring system that produces translatory motion of the clamp block at the free end with negligible rotation, so long as the frequency

is below that causing local resonances in the plates. The clamp block serves as the calibration table to which the transducer being calibrated is attached. The motion of the calibration table has no appreciable transverse or torsional components.

The loudspeaker drives the free end of the beams through a thin plastic rod glued to the speaker cone. A conventional audio oscillator and 50-watt commercial amplifier drive the speaker. Because of the resonance of the mechanical system, only a small amount of power is needed to obtain suitable vibration amplitudes at or near the resonant frequency. Furthermore, electrical waveform is not critical; the resonant system acts as a mechanical filter and gives very pure sinusoidal motion. Because the motion is sinusoidal, peak velocity and acceleration can be easily and accurately calculated from the frequency and the measured displacement.

The displacement of the calibration table is measured relative to a fixed point in space, by means of a Tuckerman optical lever strain gage and an autocollimator. This gage consists essentially of a fixed prism (connected to the reference point) and a small mirror that rotates slightly with motion of the calibration table. By observing the gage with the autocollimator, dynamic displacements ranging from 5 to 5000 microinches have been satisfactorily measured. For greater displacements, an inexpensive microscope with a built-in micrometer reticle gives good accuracy. The optical gage method has been used at NBS with apparently good results at frequencies up to 250 cps; the upper frequency limit for satisfactory measurements is believed to be determined by the mechanical inertia of the gage.

With the equipment used at NBS it was found that a single set of beams could be used for a frequency range of about three to one—from about 50 per cent to 150 per cent of the resonant frequency of the beams. To calibrate at frequencies beyond the range of one set of beams, the beams must be removed and a different set installed, a procedure requiring about 10 minutes. Only three different pairs of beams are needed to provide a calibration range of 10 to 250 cps.

Care must be taken to insure that the reference point to which the displacement measurements are referred remains motionless. At NBS this was accomplished by making the reference point part of a very heavy and rigid concrete block. An alternative would be to mount the reference point on an elastic system having a very low natural frequency, perhaps one-fifth of the lowest frequency at which calibrations are to be made. Regardless of the method of isolation used, any significant motion of the reference point during a calibration run can be detected by means of a monitoring pickup mounted on the reference-point structure.

THE FRANKLIN INSTITUTE

Minutes of Stated Meeting May 20, 1953

The Stated Monthly Meeting of The Franklin Institute was held on May 20, 1953, in the Lecture Hall. S. Wyman Rolph, President, called the meeting to order at 8:15 p.m. Approximately 390 persons attended.

The President asked for approval of the Minutes of the Stated Meeting in March, which were printed in the April JOURNAL, and of the Minutes of the Stated Meeting in April, which were printed in the May JOURNAL. There being no corrections or additions, they were approved as submitted.

The President summarized the Annual Report of the Board of Managers for 1952 and announced that the complete Report would be published in the July issue of the JOURNAL.

The Secretary stated that Mr. William Francis Gibbs, of New York City, Franklin Medalist for 1953, had been nominated by the Board of Managers for Honorary Membership in The Franklin Institute. In accordance with the provisions of the By-Laws, upon motion duly made and seconded, Mr. Gibbs was unanimously elected to Honorary Membership.

The Secretary reported the following elections to membership for the month of April:

Active	29
Associate	42
Student	17

and a total membership of 6,642 as of April 30, 1953.

The President paid tribute to Edward G. Budd, in whose memory the evening's lecture was given, under an endowment created by his widow and his son, Edward G. Budd, Jr.

The President then recognized Ralph Kelly, Chairman of the Vermilye Medal Committee of the Board of Managers, who summarized the history of the Vermilye Medal which was founded by the late William M. Vermilye, a member of the Board of Managers. He spoke of the outstanding achievements of Benjamin F. Fairless in the field of Industrial Management and cited him to receive the Vermilye Medal. The Medal and the accompanying citation were then presented to Mr. Fairless by the President with his warm congratulations.

Mr. Fairless, after accepting the Vermilye Medal, gave the Edward G. Budd Lecture on the subject of "Mid-Century Miracle."

The meeting adjourned at 9:25 p.m. with an enthusiastic vote of thanks to Mr. Fairless.

HENRY B. ALLEN
Secretary

EDITOR'S NOTE: The full proceedings of the Vermilye Medal Presentation and the text of the Edward G. Budd Lecture will be published in the August issue of the JOURNAL.

COMMITTEE ON SCIENCE AND THE ARTS

(Abstract of Proceedings of Stated Meeting Held Wednesday, May 13, 1953.)

HALL OF THE COMMITTEE,
PHILADELPHIA, MAY 13, 1953.

COLONEL HIRAM B. ELY in the Chair

The following reports were presented for final action:

No. 3253: Work of George R. Harrison.

This report recommended the award of an Elliott Cresson Medal to George Russell Harrison, of Cambridge, Massachusetts, "For obtaining and refining spectroscopic evidence bearing upon atomic structure, particularly his precision measurements in the Zeeman effect, for his outstanding improvements in the instruments of spectroscopy, and for the invaluable results obtained by these instruments."

No. 3256: Work of William F. Meggers.

This report recommended the award of an Elliott Cresson Medal to William Frederick Meggers, of Washington, D. C., "For his monumental work throughout four decades in the field of Spectroscopy,—work which has exhibited a high degree of diligence, consistent accuracy and scientific responsibility, work which has produced much of the reliable experimental data upon which rests our present detailed knowledge of the electronic structure of scores of elements."

No. 3262: Franklin Medal.

This report recommended the award of the Franklin Medal to William Francis Gibbs, of New York, New York, "Because he has labored successfully for forty years to bring the best of engineering and architecture into shipbuilding, culminating in the construction of the record-breaking S. S. 'United States'."

JOHN FRAZER,
Secretary to Committee

MEMBERSHIP

ACTIVE MEMBERS ELECTED AT THE MEETING OF THE BOARD OF MANAGERS, MAY 20, 1953

ACTIVE FAMILY

Herman F. Ball
Rae Crowther
Chester A. Milewski
Mrs. Alec Perel
Michael G. Sabia

ACTIVE

Irvin Borowsky
Paul B. Culp
William J. Erdman, II, M.D.
Marlin A. Espenshade, Ph.D.
Milan J. Fisher
Patrick D. Goggin
Ralph E. Green
William Edward Gretz
R. P. Guildford
Arthur N. Klein
Charles E. Levinson
R. Winston Liggett
David D. Mynick
Charles C. Nitchie
Paul C. Peter
Roland G. E. Ullman, Jr.
Willard York

ACTIVE NON-RESIDENT FAMILY

Robert E. Davis

NECROLOGY

William R. Armstrong '85
Thomas Blackadder '49
Mrs. J. Lloyd Coates '35
John DaGrosa '51
Mrs. James Mapes Dodge '36
A. L. Dold '42
Gano Dunn '01
Howard N. Eavenson '13
Henry L. Gowens, Jr., M.D. '44
Ruth Custis Kitchen '30
Lewis F. Moody '23
Wallace G. Murfit '51
Robert E. Naumberg '35
Thornton Oakley '36
Donald R. Perry '52
Clifford F. Simmons, M.D. '45
Mabel L. H. Thomas '34

JOURNAL OF THE FRANKLIN INSTITUTE

The following papers will appear in this JOURNAL within the next few months:

COULSON, THOMAS: The First Hundred Years of Research at The Franklin Institute.
COURT, ARNOLD: Wind Extremes as Design Factors.
ALDRICH, LELAND: Solution of Algebraic Equations.
MINORSKY, N.: On Interaction of Nonlinear Oscillators.
MARIN, JOSEPH AND JOHN A. SAUER: Plastic Stress-Strain Relations under Radial and Non-Radial Combined Stress Loading.
LANGHAAR, HENRY L.: The Principle of Complementary Energy in Nonlinear Elasticity Theory.
SWANN, W. F. G.: The Case of Charged Density Distribution *versus* Semi-Permanent Polarization as a Basis for Electret Behavior.
KU, Y. H.: A Method for Solving Third and Higher Order Nonlinear Differential Equations.
BERGER, W. J. AND EDWARD SAIBEL: On the Inversion of Continuant Matrices.
LEE, J. F.: The Thermodynamics of Critical Phenomena in Gases.

LIBRARY

The Committee on Library desires to add to the collections any technical works that members may wish to contribute. Contributions will be gratefully acknowledged and placed in the Library. Duplicates received will be transferred to other libraries as gifts of the donor.

Photostat service. Photostat prints of any material in the collections can be supplied on request.

The Library and reading room are open from 9 A. M. until 5 P. M. on Mondays, Tuesdays Thursdays and Fridays; 2 P. M. until 10 P. M. on Wednesdays; and 9 A. M. until 12 noon on Saturdays.

RECENT ADDITIONS

AERODYNAMICS

GUILLENSCHMIDT, PAUL DE. Éléments de Calcul de Construction Aéronautique. Ed. 2. 1951.

AGRICULTURE

DE ONG, ELMER RALPH. Insect, Fungus and Weed Control. 1953.

ASTRONOMY

WOOLLEY, R. V.D. R. AND STIBBS, D. W. N. The Outer Layers of a Star. 1953.

BIBLIOGRAPHY

GRAY, GEORGE J. A Bibliography of the Works of Sir Isaac Newton. 1907.
KEPLER, Johannes. Bibliographia Kepleriana. 1936.

BIOLOGICAL CHEMISTRY

Progress in Biophysics and Biophysical Chemistry. Vol. 1–2. 1950–1951.

CHEMISTRY AND CHEMICAL TECHNOLOGY

BRESCIA, FRANK. General College Chemistry. 1953.
CAUCHOIS, YVETTE. Atomes, Spectres, Matière. 1952.
DE BOER, J. H. The Dynamical Character of Adsorption. 1953.
DUVAL CLÉMENT. Inorganic Thermogravimetric Analysis. 1953.
JACOBS, MORRIS BORIS. Chemical Analysis of Industrial Solvents. 1953.
KIENTZ, LOUIS. Notions Fondamentales sur les Vernis et Peintures. Ed. 2. 1952.
KIRK, PAUL L. Crime Investigation. 1953.
MINER, CARL SHELLEY AND DALTON, N. N., ED. Glycerol. 1953.

SMITH, BURROWS ARTHUR. Semimicro Laboratory Exercises in General Chemistry. Ed. 2. 1952.
WHISTLER, ROY LESTER AND SMART, CHARLES LEWIS. Polysaccharide Chemistry. 1953.

CIVIL ENGINEERING

CAIN WILLIAM. Practical Designing of Retaining Walls. Ed. 6. 1910.
CAIN, WILLIAM. Theory of Voussoir Arches. Ed. 3. 1905.

DICTIONARIES

WHITE, FRANCIS SELLON. A History of Inventions and Discoveries: Alphabetically Arranged. 1827.

ELECTRONICS

SOMMER, A. Photoelectric Tubes. Ed. 2. 1951.

ENGINEERING

MARSTON, ANSON; WINFREY, ROBLEY AND HEMPSTEAD, JEAN C. Engineering Valuation and Depreciation. Ed. 2. 1953.

GENERAL

CORNOG, WILLIAM H. School of the Republic, 1893–1943. 1952.
FREY, CARROLL. The Independence Square Neighborhood. 1926.
KOHLER, FRED. Evolution and Human Destiny. 1952.
ZIMMERMAN, OSWALD THEODORE AND LAVINE, IRVIN. Handbook of Material Trade Names. 1953.

GEOLOGY

JEFFREYS, HAROLD. Earth; its Origin, History and Constitution. Ed. 3. 1952.
RUSSELL, R. C. H. AND MACMILLAN, D. H. Waves and Tides. 1953.

HISTORY

BROWNE, CHARLES ALBERT AND WEEKS, MARY ELVIRA. A History of the American Chemical Society. 1952.

MANUFACTURE

BILFINGER, ROBERT. Handbuch der Elektrolytischen Verchromung. Ed. 2. 1952.
LEUCHS, JOHANN CARL. Beschreibung und Abbildung der Verbesserten Amerikanischen Malmülen. 1828.
SCHALLER, GILBERT S. Engineering Manufacturing Methods. 1953.
STIERI, EMANUELE. Basic Welding Principles. 1953.

MARINE ENGINEERING

LA DAGE, JOHN H. Modern Ships. 1953.

MATHEMATICS

DOOB, JOSEPH L. Stochastic Processes. 1953.
FREYTAG-LÖRINGHOFF, BRUNO. Philosophical Problems of Mathematics. 1951.
MILNE, WILLIAM EDMUND. Numerical Solution of Differential Equations. 1953.
QUENOUILLE, M. H. Associated Measurements. 1952.
ROSSER, JOHN BARKLEY. Logic for Mathematicians. 1953.
THRON, WOLFGANG J. Introduction to the Theory of Functions of a Complex Variable. 1953.
U. S. BUREAU OF STANDARDS. Construction and Applications of Conformal Maps. 1952.
U. S. BUREAU OF STANDARDS. Tables of Chebyshev Polynomials $S_n(x)$ and $C_n(x)$. 1952.
WOLD, HERMAN AND JURÉEN, LARS. Demand Analysis. 1953.

MECHANICAL ENGINEERING

CERCONE DE LUCIA, C. AND MIRONE, T. Elementi di Technologia Meccanica. Ed. 13. 1953.
FONTAINE, ROBERT. Formulaire de Construction Mécanique. Ed. 4. 1951.
PAPIN, DENIS. Traite de Plusieurs Nouvelles Machines et Inventions Extraordinaires sur Differents Sujets. 1698.

OPTICS

SCHAEFFER, HAROLD F. Microscopy for Chemists. 1953.

PHOTOGRAPHY

DESCHIN, JACOB. Rollei Photography. 1952.

PHYSICS

BÉGHIN, HENRI. Cours de Mécanique Théorique et Appliquée. Vol. 1–2. 1952.
BROGLIE, PRINCE LOUIS VICTOR DE. La Theorie des Particles de Spin ½ (Électrons de Dirac). 1952.
CHAMBADAL, PAUL. Les Machines Thermique. 1952.
CHASTELET GABRIELLE ÉMILIE. Institutions de Physique. 1740.
ESNAULT-PELTERIE, ROBERT. Dimensional Analysis. 1952.
FLETCHER, HARVEY. Speech and Hearing in Communication. Ed. 2. 1953.
GASTON, IGNACE. La Statique ou la Science des Forces Mouvantes. Ed. 3. 1688.
GERLACH, FRIEDRICH WILHELM ANTON. Kleine Mechanische Weisheit. 1763.
HENNIG, GERHARD. Dauermagnettechnik. 1952.
HUGHES, DONALD L. Pile Neutron Research. 1953.
JAYNES, E. T. Ferroelectricity. 1953.
MANDEY, VENTERUS AND MOXON, JAMES. Mechanik-Powers. 1696.
POTTER, J. H. Technical Calculations in Heat Engineering. 1952.
U. S. BUREAU OF STANDARDS. Mass Spectroscopy in Physics Research. 1953.
WALLIS, JOHN. A Discourse of Gravity and Gravitation. 1675.
WALLIS, JOHN. Mechanica. 1670.

PLASTICS

THINIUS, KURT. Analystische Chemie der Plaste. 1952.

RAILROADS

NORDMANN, HANS. Die Mechanik der Zugförderung und ihre Wissenschaftlich Experimentellen Grundlagen. 1952.

SCIENCE

BATE, JOHN. The Mysteries of Nature and Art. Ed. 2. 1635.
HERO, of Alexandria. Spiritalium Liber. 1575.
KNIGHT, GOWIN. An Attempt to Demonstrate that All the Phaenomena in Nature may be Explained by Two Simple Active Principles. 1754.
MONTAGUE, M. F. ASHLEY, ED. Studies and Addresses in the History of Science. 1953.
PANCIROLI, GUDIO. Nova Reperta Sive Rerum Memorabilium Recens Inventarum. Ed. 2. 1608.
PANCIROLI, GUDIO. Rerum Memorabilium iam Olim Deperditarum & Contra Recens Atque Ingeniose Inventarum. 1599.
POWELL, [THOMAS]. Humane Industry. 1661.
SARTON, GEORGE. A Guide to the History of Science. 1952.

SCIENTIFIC INSTRUMENTS

POLENI, GIOVANNI. Miscellanea. 1709.

TEXTILES

AMERICAN VISCOSE CORP. Rayon Technology. Ed. 2. 1953.
RATH, HERMANN. Lehrbuch der Textilchemie. 1952.

Machine for Shrinking Speeches.—A device for stretching or contracting the length of a speech without affecting voice quality was demonstrated at the closing session of the 1953 convention of the Institute of Radio Engineers, in New York.

The speech stretcher-compressor was described in a paper prepared by Grant Fairbanks, W. L. Everitt and R. P. Jaeger of the University of Illinois, who reported that by means of a new and original device the time consumed by a recorded speech can, in a playback, be increased or decreased and that this can be done without the distortions usually accompanying such a change in speed, when normal voices turn to screeches when speeded up or to growls when slowed down.

In addition to such practical considerations as enabling broadcasters to tailor a political speech, for example, to fit a time period without changing the speaker's words or his method of delivery, the device has military advantages by shortening the time of message transmission and, at the other end, lengthening the time can aid linguists and students in analyzing speech patterns.

Color Television.—One of the prime problems of bringing color television to the public is the development of a system which will require a minimum of space on the television spectrum. This is important because the less spectrum space required for each station, the more stations will be able to broadcast.

In line with this, R. D. Kell and A. C. Schroeder of RCA Laboratories presented a technical paper at the recent I. R. E. convention called "Optimum Utilization of the Radio Frequency Channel for Color TV." They pointed out that in order to produce a TV image in color, three communication signals must be available. The first of these may be used to transmit the scene brightness, the second the degree of color saturation and the third the hue or color. In order to provide this information and still keep the color system compatible with present black and white, a subcarrier is introduced on the main carrier. In essence, the color information rides "piggy back" on the signals of a standard black and white system.

Rubber Gaskets for Steel Storage Tanks.—Bolted steel storage tanks used by Army Engineers are now effectively sealed from leakage with new extra soft gaskets. The rubber gasketing compound was developed by the Thermoid Corporation of Trenton, N. J., in cooperation with the Engineer Research & Development Laboratories of Ft. Belvoir, Va.

Special filet radii bending gaskets are used to seal the horizontal seams of the steel tanks at points of stave overlap. Chime gaskets seal corner joints and those areas at which three or more plates or segments overlap. The tanks, ranging from 100 to 10,000 barrel (420,000 gallons) capacities, are used in rear and forward combat areas as well as at advance air fields. They are shipped knocked-down for field erection and are designed for easy assembly.

The Hycar rubber gaskets are not affected by a variety of liquids including high octane gasoline, aromatic or aliphatic fuels, lubricating oils, or even drinking water. The seals can be used in the Arctic and the Tropics. They have excellent resistance to the deteriorating effects of petroleum products and chemicals, extreme temperatures and pressures. The compound used in them is non-toxic and will not contaminate drinking water. The new gaskets are only half the hardness of those previously used.

NOTES FROM THE BIOCHEMICAL RESEARCH FOUNDATION

ELLICE McDONALD, DIRECTOR

CANCER CELL DIVISION

FOREWORD

BY

Ellice McDonald, Director

The following two papers are a continuation of our study of the use of chemical analogs of the substrate glucose, so important in the metabolism of cancer, in order to evaluate their importance in the restriction of cancer cell division. This is an attempt to put the screening of cancer inhibiting compounds on a scientific basis instead of the present experimental and error type of effort.

Two tests have been used, the measurement of inhibition of carbohydrate metabolism of the yeast cell and the inhibition of cell division measured by the mitotic index in tissue culture. The usual medium of tissue culture contains 1 gm. of glucose in 1000 ml. and the material to be tested is added or substituted for this. Mitotic index is much more reliable than direct measurement of the growing culture as this last is difficult to measure with accuracy. As will be seen glucosamine is much less potent in inhibition than 2-desoxy-D-glucose (2DG). Most of these tests were done before the publication of Quastel's paper in *Nature* (J. H. QUASTEL and A. CANTERO, "Inhibition of Tumor Growth by D-Glucosamine," *Nature*, **171**, 252 (1953)).

D-GLUCOSAMINE AS AN ANTAGONIST OF GLUCOSE IN CARBOHYDRATE METABOLISM OF YEAST

BY

GLADYS E. WOODWARD AND MARIE T. HUDSON

A program aimed at the control of cancer through the inhibition of carbohydrate metabolism by the use of carbohydrate analogs as antagonists has been under way at this laboratory (1). It has already been shown that 2-desoxy-D-glucose (2DG), an analog of glucose, strongly inhibits the anaerobic and aerobic fermentation of glucose by yeast (2, 3), the anaerobic glycolysis of glucose by surviving tumor slices (4), the growth of yeast cultures (3) and the growth of embryonic chicken-heart fibroblasts (5). The inhibition was shown to be of the competitive type in the case of anaerobic fermentation and growth of yeast.

In the hope of finding inhibitors more potent than 2DG, numerous other carbohydrate analogs have been investigated with the yeast cell as the test system. Although most of these compounds had no effect whatsoever on the metabolism of yeast, it was found that D-glucosamine, an analog of glucose which like 2DG differed structurally from glucose only at the number 2 carbon atom, possessed inhibitory power similar to that of 2DG. Anaerobic fermentation and growth of yeast were inhibited, while respiration was hardly affected. The degree of inhibition produced by glucosamine was, however, somewhat less than that produced by 2DG under comparable conditions. Glucosamine, in contrast to 2DG, did not produce any inhibition of fermentation aerobically under conditions where 2DG produced extensive inhibition.

INHIBITION OF ANAEROBIC FERMENTATION BY D-GLUCOSAMINE

The rate of fermentation was measured in a Warburg manometric apparatus at 37° C. in mixtures consisting of 4 mg. pressed bakers' yeast (Fleischmann's starch-free) and the desired amounts of glucose or of glucose plus glucosamine hydrochloride in a total volume of 2 ml. of water. It was not considered necessary to neutralize the glucosamine hydrochloride since fermentation occurs naturally under acid conditions. The gas phase was 5 per cent CO_2 in N_2.

The degree of inhibition of rate of fermentation produced by increasing proportions of glucosamine to glucose is shown in Table I. It is apparent that the degree of inhibition is dependent upon the ratio of glucosamine to glucose rather than upon the absolute amount of glucosamine present. This observation suggests that the inhibition is due to competition between glucose and glucosamine for an enzyme which determines the rate of fermentation.

Further evidence for the competitive nature of the inhibition is given in Fig. 1, where the test for competitive inhibition described by

TABLE I.—*D-Glucosamine Inhibition of the Anaerobic Fermentation of Glucose by Yeast.*

Glucosamine: Glucose (Molar Ratio)	Glucosamine Concentration M	Glucose Concentration M	Inhibition %
0.2 :1	0.01	0.05	19
0.4 :1	0.02	0.05	26
0.5 :1	0.01	0.02	35
1 :1	0.02	0.02	44
1 :1	0.05	0.05	40
2 :1	0.04	0.02	51
5 :1	0.10	0.02	57
5 :1	0.25	0.05	53

Lineweaver and Burk (6) is applied. Reciprocals of the initial fermentation rates obtained with and without 0.01 M glucosamine at several levels of glucose concentration were plotted against reciprocals of the glucose concentrations. Straight lines resulted with approximately the same intercept on the $1/V$ axis and an increased slope for the line depicting the inhibited reaction. This picture fits the criteria of Lineweaver and Burk for competitive inhibition.

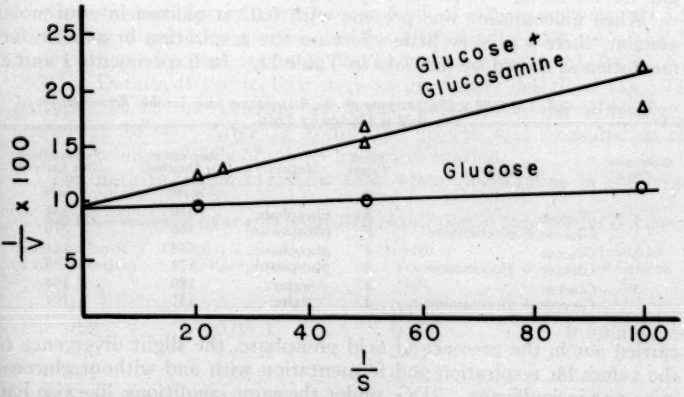


FIG. 1. Competitive inhibition by D-glucosamine in the anaerobic fermentation of glucose by yeast. Concentration of inhibitor, 0.01 M.

V = initial rate of fermentation expressed as µl. CO_2 per min.
S = initial substrate concentration in moles per liter.

Calculation of the enzyme-substrate and enzyme-inhibitor dissociation constants, K_S and K_I, respectively, were made from the data in Fig. 1 according to the Lineweaver and Burk equation (6). K_S was found to be 2.3×10^{-3} and K_I, 2.0×10^{-3}. Since the dissociation

constants are almost the same, it appears that glucosamine and glucose have approximately the same affinity for the enzyme concerned. By the same means it was previously shown (2) that the affinity of 2DG for the eyzyme was about 4 times as great as the affinity of glucose for the enzyme. The potency of glucosamine as an inhibitor of anaerobic fermentation is, therefore, only about 25 per cent that of 2DG.

D-Glucosamine has been reported by Harpur and Quastel (7) to be a competitive inhibitor of glucose phosphorylation by brain hexokinase. Indication was obtained by Grant and Long (8) that it was a competitive inhibitor also of yeast hexokinase.

EFFECT OF D-GLUCOSAMINE ON RESPIRATION AND AEROBIC FERMENTATION

Fermentation and respiration studies were carried out in the Warburg apparatus at 37° C. in air. KOH was present in the center well of a flask to absorb the CO_2 produced when the rate of O_2-uptake was the measurement desired. Reaction mixtures consisted of 4 or 8 mg. of pressed yeast and glucose or glucose plus glucosamine hydrochloride in 2 ml. of 0.02 M KH_2PO_4 or water. The CO_2 from fermentation was calculated as the CO_2 in excess of that produced by oxidation of the glucose, assuming a respiratory quotient of 1.

When glucosamine was present with 0.02 M glucose in equi-molar amount, there was very little effect on the respiration or aerobic fermentation as shown by the data in Table II. In Experiments 1 and 2,

TABLE II.—*Effect of 0.02 M Glucosamine on the Respiration and Aerobic Fermentation of 0.02 M Glucose by Yeast.*

Experiment No.	Substrate	Yeast Weight	Medium	Respiration (O_2 Absorbed)	Fermentation (CO_2 Evolved)
		mg.		µl./hr.	µl./hr.
1	Glucose	8	phosphate	379	338
	Glucose + glucosamine	8	phosphate	356	370
2	Glucose	4	phosphate	132	134
	Glucose + glucosamine	4	phosphate	124	138
3	Glucose	4	water	189	104
	Glucose + glucosamine	4	water	151	101

carried out in the presence of acid phosphate, the slight divergence of the values for respiration and fermentation with and without glucosamine was insignificant. 2DG, under the same conditions, likewise had been found not to inhibit respiration, but on the contrary to inhibit the aerobic fermentation 75 per cent (3), about the same degree of inhibition as was produced anaerobically either in the presence or absence of phosphate (2). In Experiment 3, carried out in the absence of added phosphate in order to make the conditions for aerobic and anaerobic fermentation exactly comparable, there was still no inhibition of aerobic fermentation by glucosamine, but a possible 20 per cent inhibition of respiration. With 2DG in the absence of added phosphate,

there was 84 per cent inhibition of aerobic fermentation, about the same degree of inhibition found in the presence of phosphate. In equal concentrations, therefore, glucosamine was extremely different from 2DG in that no inhibition of fermentation was produced aerobically by glucosamine while extensive inhibition was produced by 2DG.

The effects of glucosamine and 2DG on aerobic fermentation were compared also at concentrations at which equal inhibition of anaerobic fermentation occurred, that is, about 44 per cent. 2DG in concentrations which produced 54 per cent and 37 per cent inhibition of the anaerobic fermentation with 0.02 M glucose produced 52 per cent and 32 per cent, respectively, of the aerobic fermentation. With 2DG, therefore, approximately the same degree of inhibition was obtained aerobically as anaerobically regardless of the level of inhibition. With glucosamine at a concentration which produced 44 per cent inhibition anaerobically, there was no inhibition aerobically. There appears to be a fundamental difference, therefore, between the effects of glucosamine and 2DG on aerobic fermentation. Further experimentation will be required to explain this difference.

INHIBITION OF GROWTH BY D-GLUCOSAMINE

Growth studies were carried out in 6 ml. of culture medium (pH 5.5) containing 2 per cent glucose and inoculated with 0.04 ml. of an 18 hour culture of a pure strain of *Saccharomyces cervisiae* (A.T.C.C. No. 7921). Details of the technic were as previously described (3). The cultures, in 25 ml. Erlenmeyer flasks, were shaken on a horizontal shaker at 30° C. After 27 hours, the growth was measured as the packed volume of cells obtained by centrifugation.

The data in Table III show that when glucosamine in the larger

TABLE III.—*Inhibitory Effect of D-Glucosamine on the Growth of Yeast in 2 Per Cent Glucose.*

Glucosamine-HCl Concentration %	Glucosamine: Glucose (Molar Ratio)	Growth of Yeast ml.	Inhibition of Growth %
0		0.028	
0.2	0.08:1	0.034	0 (stimulated)
2.0	0.84:1	0.017	39
4.0	1.68:1	0.003	89

amounts was incorporated in the culture medium, extensive inhibition of yeast growth occurred. A small amount of glucosamine appeared to stimulate growth slightly. The relative potency of glucosamine and 2DG as inhibitors of yeast growth is shown by the amounts of each inhibitor required to produce 50 per cent inhibition of growth; with glucosamine approximately 1 molar equivalent is required, while with 2DG only 0.05 molar equivalent is required (3). Glucosamine is, therefore, only about 5 per cent as effective as 2DG for inhibition of yeast

growth, a figure about the same order of magnitude as found by Ely *et al.* (9) for inhibition of growth of embryonic chicken-heart fibroblasts.

SUMMARY

D-Glucosamine, an analog of glucose, has been found to inhibit the anaerobic fermentation and growth of yeast in the presence of glucose. Aerobic fermentation and respiration were but little affected. D-Glucosamine was about 25 per cent as potent as 2-desoxy-D-glucose as an inhibitor of anaerobic fermentation and about 5 per cent as potent as an inhibitor of growth.

REFERENCES

(1) E. McDONALD AND F. B. CRAMER, JOUR. FRANKLIN INST., **253**, 351 (1952).
(2) F. B. CRAMER AND G. E. WOODWARD, JOUR. FRANKLIN INST., **253**, 354 (1952).
(3) G. E. WOODWARD, JOUR. FRANKLIN INST., **254**, 553 (1952).
(4) G. E. WOODWARD AND F. B. CRAMER, JOUR. FRANKLIN INST., **254**, 259 (1952).
(5) J. O. ELY, F. A. TULL AND J. A. HARD, JOUR. FRANKLIN INST., **253**, 361 (1952).
(6) H. LINEWEAVER AND D. BURK, *J. Am. Chem. Soc.*, **56**, 658 (1934).
(7) R. P. HARPUR AND J. H. QUASTEL, *Nature*, **164**, 693 (1949).
(8) P. T. GRANT AND C. LONG, *Biochem. J.*, **50**, xx (1952).
(9) J. O. ELY, F. A. TULL AND J. M. SCHANEN, JOUR. FRANKLIN INST., **255**, 559 (1953).

THE EFFECT OF GLUCOSAMINE ON CULTURE OF EMBRYO CHICKEN HEART CELLS

BY

J. O. ELY, F. A. TULL AND J. M. SCHANEN

Glucosamine hydrochloride incorporated in the tissue culture medium of chicken heart cells was found to depress the growth of the cultures, and to alter the growth characteristics.

METHODS

Cultures of 8 to 10 day old chicken embryo heart were grown on glass coverslips. The culture medium consisted of one part Tyrode solution (without sugar in some instances), one part chicken embryo extract, and two parts of chicken plasma. Sugars were incorporated in the Tyrode solution in amounts necessary to give the concentrations desired for the cultures. Cultures were incubated at 37° C. for 66 hours.

RESULTS

Glucosamine as the only sugar added to the medium, or with added glucose, produced a growth (Fig. 1A) which appeared to be as great as

FIG. 1. Influence of glucosamine on tissue cultures of chicken heart cells.

A. 66 hour culture with 0.2 per cent glucosamine
B. 66 hour culture with no sugar
C. 66 hour culture with 0.1 per cent glucose

the growth with no sugar present (Fig. 1B). The cultures were characterized by coarseness and great thickness of the growth area compared to the cultures with no sugar or with only glucose present (Fig. 1C). The mitotic index, Table I, of these cultures with glucosamine in the medium was less than that of the cultures in which there was no sugar in the medium. The mitotic index (Table I) of the cultures with 0.1 per cent glucosamine as the only added sugar was less than that of the cultures without sugar in the medium. As the amount of glucosamine was increased in the medium already containing 0.1 per cent glucose the mitotic index of the cultures decreased; this indicates that glucosamine is a mitotic depressant.

The potency of glucosamine as a mitotic depressant is not great compared to 2-desoxyglucose (2DG). 0.3 per cent glucosamine in a medium containing 0.1 per cent glucose reduced the mitotic index from 2.48 (glucose alone) to 0.81 whereas 0.01 per cent 2DG reduced the index to 0.58. This indicates that glucosamine may be only about 3

TABLE I.—*Mitotic Indexes.*

Concentration of Sugars in Media %		Mitotic Index
Glucosamine	0.1	1.16
Glucosamine	0.1	
Glucose	0.1	1.72
Glucosamine	0.2	
Glucose	0.1	1.20
No sugar		1.84
Glucose	0.1	2.48
Glucosamine	0.3	
Glucose	0.1	0.81
Glucosamine	0.2	
Glucose	0.1	
2DG	0.005	0.84
Glucose	0.1	
2DG	0.005	1.30
Glucosamine	0.2	
Glucose	0.1	
2DG	0.01	0.66
Glucose	0.1	
2DG	0.01	0.58

per cent as effective as is 2DG. A concentration as low as 0.005 per cent 2DG reduced the mitotic index approximately 50 per cent.

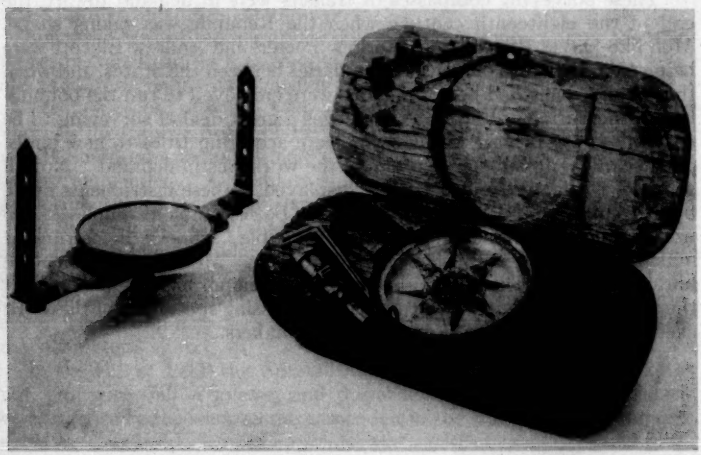
The reduction of the mitotic index and the apparent small reduction in the gross size of the cultures indicates that glucosamine depresses the growth of embryo chicken heart cells in tissue cultures.

V.2
195

NOTES FROM THE FRANKLIN INSTITUTE MUSEUM

SURVEYOR'S COMPASS

Benjamin Rittenhouse has never been as well known as his elder brother David in the field of instrument making, but his work shows him to have been a craftsman of high accomplishment, undeserving the obscurity into which he has fallen. After receiving an education at Norristown, Montgomery County, Benjamin assisted David in making instruments before taking part in the Revolutionary War. Wounded at the battle of Brandywine, he had the misfortune to fall into enemy hands. However this was not the end of his service for he re-appears



Surveyor's Compass. Benjamin Rittenhouse's on right, Thomas Whitney's on left.

in February, 1778, as Superintendent of the gun-lock factory which had been set up in Philadelphia, proof that his mechanical ability had already found recognition among his contemporaries.

Unfortunately it is not always possible to distinguish which of the two brothers made some of the instruments bearing the Rittenhouse name, but there are several grandfather clocks in existence which can be identified as having been the work of Benjamin, who appears in the Philadelphia Directory as a mathematical instrument maker located at Pine Street above 12th in Philadelphia. That he was a craftsman

of undoubted skill is demonstrated by the beautiful workmanship displayed in a surveying compass bearing his name, which has been presented to the Institute Museum by Mr. and Mrs. Victor F. Emerson, of Villanova.

Not the least interesting feature of the gift is the case in which the instrument is enclosed. This has been cut from a single piece of wood, carefully cut away so that when the instrument is disassembled, its parts laid in their allotted places, and the lid is closed, everything fits snugly and the instrument can be transported without fear of injury to its parts.

The Museum has a similar instrument made by Thomas Whitney in 1812, which was presented by Mr. Harold E. Gillingham, of Germantown. Whitney had come from London and had established himself at 74 South Front Street, Philadelphia.

These surveying compasses or transits were in demand toward the end of the eighteenth century when the Republic was taking shape. Men like David Rittenhouse, Andrew Porter, and Andrew Ellicott were being called upon to define the boundaries between the States, and when these major lines had been run, they were employed to run the boundaries of the various counties. There was a great deal of surveying to be done for those private owners who were acquiring titles to new lands, and technical apparatus to promote the work was in demand. Not all of it was of the fine craftsmanship displayed in these instruments made by Rittenhouse and Whitney, which are both notable examples of the work issuing from the shops of a remarkable "school" of instrument makers who were established in Philadelphia. Most of these men are known only by their advertisements in contemporary newspapers, but the Institute Museum is fortunate in possessing specimens of the work executed by several of the better known makers.

NOTES FROM THE FRANKLIN INSTITUTE LABORATORIES FOR RESEARCH AND DEVELOPMENT

NICOL H. SMITH, DIRECTOR

THE ENGINEERING DEVELOPMENT OF THE 280-MM. WEAPON*

BY

AXEL O. BERGHOLM

Members of the American Ordnance Association and Honored Guests: It is my function in this program to discuss the engineering development of the 280-mm. weapon, better known as the atomic gun, in the limited time of a few minutes.

Since the task of talking about this weapon in a few minutes is just impossible, all I can do is to highlight its history, some salient features and add a few comments regarding the future.

The first record of which I am aware regarding this weapon is in the report of the Caliber Board of 1919. In this report, the characteristics of a 240-mm. gun, including 360° traverse, muzzle velocity and range, were recommended. As in many other projects, interest in this weapon lay dormant until revived during World War II.

In the early part of World War II, Drs. Henry B. Allen and Rupen Eksergian of The Franklin Institute cooperated with General Gideon Barnes in organizing an Ordnance Design Office at The Franklin Institute to exploit the engineering personnel available in the Philadelphia area. This group became a Sub-Office of the Office, Chief of Ordnance in Washington.

In the Fall of 1944, this Sub-Office was instructed to develop a 240-mm. gun with 360° traverse, and the other characteristics stipulated were practically the same as recommended in the Caliber Board Report of 1919.

During World War I, a large caliber gun, using a double recoil carriage, was designed, constructed and successfully tested. This gun, largely the work of Dr. Eksergian who was, at that time, an officer in the Ordnance Department, had become dormant after the close of World War I. This previous design indicated however, that with the use of a double recoil system, the horizontal ground force, during firing, could be made relatively small.

Consequently, it was immediately decided to incorporate a double recoil system into the 240-mm. gun in order to gain the desired mobility required in modern fluid warfare.

* Short talk presented to members of the Philadelphia Chapter of the American Ordnance Association at a meeting held in The Franklin Institute, March 10, 1953.

In the beginning, the design studies were carried on at a low level, but in 1945, design effort was expanded. The entire design studies were very comprehensive. The original design was for a 240-mm. gun carriage with a 240-mm. gun and 280-mm. Howitzer. This dual purpose carriage was later discarded because of difficulties in using the gun recoil mechanism for the Howitzer. Numerous arrangements of recoiling parts, various types of recoil mechanisms, eight methods of gun-tube design, several types of prime movers, and several methods of subdividing the weapon into separate transport loads were all considered.

The manufacturing drawings for the 240-mm. carriage and gun were delivered to Watertown Arsenal in April 1949. The weapon was then constructed and tested, and the final outcome is the weapon you saw today.

Several times in early 1950, the Ordnance Department asked our opinion on increasing the weight of the projectile, still maintaining the same muzzle velocity. This was like throwing a bomb into the camp! I promptly stated it could not be done, and was informed that I was a conservative, old man. I heartily disagreed again!

A few comments are in order concerning the salient features of this weapon. To the Ordnance Engineer, the fact that the gun fires an atomic projectile is not particularly significant. The mobility of the weapon is the important factor, and particularly so in view of modern fluid warfare. Now, several things have made this possible. One is the advances that have been made in modern automotive power. Another is the use of the double recoil system. Still another valuable feature is that the weapon can be traversed 360°.

For the benefit of those who are not acquainted with a double recoil system, I shall endeavor to give a brief description of it. The weapon has two independent recoil mechanisms. One connects the gun to the top carriage. The second mechanism connects the top carriage to the bottom carriage. This has many advantages but the primary advantage is that over 90 per cent of the weight of the weapon can be used to absorb the recoil force. This reduced the 4,000,000-lb. powder-gas force to a 100,000-lb. horizontal ground reaction allowing the weapon to be placed directly on the ground and fired without trails, thus eliminating any digging in and saving valuable emplacement time. The weapon can be emplaced and fired within 15 minutes, as compared to 6 to 8 hours for the 8-in. gun M1. There are many other interesting features and engineering problems that could be discussed at length. There seems to be no end to problems being encountered.

The design and development work on the 280-mm. gun has been a most stimulating experience for us at The Franklin Institute. Very helpful in attaining success has been the degree of freedom that the Ordnance Department has granted us in performing our work. There have been several drawbacks, however, principally the lack of Ordnance

V.2
195

experience by some of us at the beginning of World War II and the lack of readily available ordnance engineering data. It is about this last item that I want to say a few words.

For my part, there is much to be learned. Data must be acquired on the physical properties of steel under dynamic loading conditions experienced in artillery in normal and cold weather operation, and on the physical properties of soils under dynamic loading. A thorough knowledge of dynamic interactions between the components of a weapon and between the weapon and the soil can only be attained by a comprehensive, planned engineering research program. With this knowledge, engineering procedures for Ordnance Design can be specified. At the present time, many Ordnance Engineering personnel are fast approaching the retirement age. With them, goes their experience, knowledge and their little black books. The next in line have to start from scratch, as so many of us did in World War II.

If a comprehensive Research Plan is adopted for Ordnance Design, a real contribution will be made to the National Defense.

ENGINEERING EVALUATION OF FLIGHT TESTING DIVE OR TOSS BOMB COMPUTERS

BY

D. B. HOUGHTON

Recent developments in jet aircraft have forced corresponding developments in fire control systems to aid the pilot of a single-seater bomber to drop his ammunition "on target." The pilot's estimate of the release point is not sufficiently reliable. Not only is he too busy navigating his plane, but the extremes of speed and altitude of modern aircraft make this task too exacting for a human and dictate the necessity of an automatic system.

Then too, recent developments in anti-aircraft equipment cause the pilot to resort to relatively unconventional bombing approaches and sometimes severe maneuvers during the approach. The modern fire control system enables the pilot to concentrate on tracking the target while it measures the necessary quantities such as altitude and airspeed, computes the proper release point and releases the bomb automatically when such a point is reached. Evaluation of a system of this type is a tedious procedure.

The system is subject to a multitude of errors including the pilot's tracking error, the error in the input measuring instruments, errors due to assumptions and approximations in the theory on which the computer is based, errors of mechanization in the computer and many more. To perfect a system of this type, one must do more than just make a series of flight tests to determine by how much the bomb misses

the target. The developer wants to know what is causing the over-all error so that he can concentrate on improving the weakest part of the system.

The Analysis Section of The Franklin Institute Laboratories has developed a method of engineering evaluation of flight testing bombing systems of this type. Particular emphasis has been given to bombs since previous work was mainly concerned with guns and rockets. The method includes straight and level bombing, but was primarily developed for evaluating systems in which the pilot dives at the target and then pulls up, the bomb being released either in the dive or pull-up. These are known as dive and toss bombing, respectively.

Through precise instrumentation of the aircraft and a target range, basic data are recorded continuously during the dive approach and the pull-out. This information is tabulated at periodic intervals. Formulas were derived, which, with the use of a large-scale general purpose digital calculating machine, enable one to calculate the required information from the basic data and previously computed quantities.

In this way the various component leads due to a wind, target motion and the effect of gravity are obtained. Both existing leads as well as those that are required for a hit are obtained. The method also makes available considerable additional information such as the aircraft's angles of attack and skid, its angular rates and its accelerations. Use of the method is expected to enable the developer to perfect such systems faster and furnish the Armed Services with improved equipment that is vitally needed in the defense of our country.

NOTES FROM THE BARTOL RESEARCH FOUNDATION

W. F. G. SWANN, DIRECTOR

ELECTRICAL CONDUCTION AND THERMOELECTRIC POWER IN THORIUM OXIDE CRYSTALS

BY

O. A. WEINREICH AND W. E. DANFORTH

The conductivity and thermoelectric power of homogeneous crystalline specimens of thorium oxide are being measured in different gas atmospheres and in vacuum. Conduction in hydrogen is relatively stable and is accompanied by electrolytic discoloration. In oxygen, conductivity is higher, increases markedly with flow of current and is not accompanied by visibly electrolytic deposits.

In all cases where a change is made from an oxygen atmosphere to a non-oxygen atmosphere (or vacuum) the conductivity decreases. One therefore speculates that the material is a p-type conductor when in oxygen and in the temperature range concerned (700°–1000° C.) This is corroborated by the polarity of the Seebeck effect. In vacuum at higher temperatures evidence points to n-type conduction.

Data regarding the variation of thermoelectric power with temperature in an oxygen atmosphere yield energy-gap values in fair agreement with those obtained from electrical conductivity. Using the simplest semi-conductor theory with present data the calculated density of impurity centers is unreasonably high. However, it is to be expected that, under the circumstances, the impurity density will vary with temperature. An attempt to take this into account mathematically is in progress.

THE CASE OF CHARGED DENSITY DISTRIBUTION *versus* SEMI-PERMANENT POLARIZATION AS A BASIS FOR ELECTRET BEHAVIOR

BY

W. F. G. SWANN

It is generally recognized that a distribution of volume and surface charges of total amount zero is equivalent, as regards field produced, to a suitably chosen distribution of polarization. A formal proof of this matter is given. The distribution of polarization lacks uniqueness to the extent of any contribution which, considered as a vector, is not derivable from a potential.

It is shown that, if an electret were composed of ordinary charge distributions, its effects would diminish by fifty per cent in a day or two on the basis of any reasonable assumption for the medium conductivity. Electret behavior must thus be based upon a semi-permanent polarization decaying in a manner independent of conductivity. The effects observed can be regarded as arising from lag in the action of conductivity in providing for cancellation of the effects of polarization at each instant.

The fact that the polarization, regarded as a sub-macroscopic distribution of charges, is itself not annihilated by the conductivity with a time constant appropriate thereto calls for explanation, and such is given.

FUNDAMENTALS IN THE BEHAVIOR OF ELECTRETS

BY

W. F. G. SWANN

The behavior of the dielectric in the case of an exponential decay is determined by the initial polarization P_0, which may vary across the electret, the initial potential difference V_0 between the surfaces of the electret, the coefficient, α, in the exponential decay, and the quantity RC representing the product of the resistance and capacity associated with the electret.

In the present paper it is demonstrated that, in the case of an electret which shows a maximum in the curve representing potential difference between surfaces plotted against time, all of the foregoing quantities may be obtained from measurements of the initial potential difference, the maximum potential difference, the time for the potential difference to fall to zero, and the time for it to attain its maximum. The theory of the matter is extended to the case where the semi-permanent polarization is represented by a finite or infinite number of contributions, each having its own particular relaxation time.

The paper discusses the question of what determines the initial potential difference after the electret is removed from its forming plates. It is shown that a reasonable understanding of the facts can be obtained qualitatively and quantitatively by assuming that on separating the plates slightly from the surfaces of the electret the charge densities retained by the plates are limited by the condition that the resulting field in the air space is less than the breakdown field in air.

THE STORAGE OF ENERGY IN SILVER ACTIVATED POTASSIUM CHLORIDE

BY

C. E. MANDEVILLE AND H. O. ALBRECHT

When KCl-Ag is irradiated by nuclear particles such as alpha, beta, or gamma rays, it fluoresces and phosphoresces with the emission

of two bands of light, one centered at 2800 Å, the other at 4350 Å. The behavior of the ultraviolet band under photostimulation has been used to measure the energy storage in the irradiated crystals of KCl-Ag. A specimen of KCl-Ag weighing about one gram was placed in a light-tight container after having received a dosage of approximately three roentgens. After twenty-four hours, the irradiated specimen was photostimulated by a one watt tungsten lamp, located seven centimeters from the crystal. The counting rate in a photosensitive Geiger counter nearby increased from nothing detectable above cosmic ray background to 10^5 counts per minute, while the stimulating light was on. During one minute of photostimulation, the counting rate decreased to about 2.5×10^4 counts per minute and dropped to about 40 counts per minute when the stimulating light was extinguished at the end of one minute. The first two above-mentioned counting rates were recorded while the stimulating light was irradiating the crystal of KCl-Ag which had been excited 24 hours previously. They arose, of course, from the release of "stored ultraviolet" from the piece of KCl-Ag. The Geiger counter gives no response to the visible and near-visible stimulating light, making it possible to observe the stimulated emission while the stimulating light is on.

REPORT ON THE PRESENT STATUS OF THE HIGH ENERGY VAN DE GRAAFF GENERATOR CONSTRUCTED AT THE BARTOL RESEARCH FOUNDATION OF THE FRANKLIN INSTITUTE

BY

C. P. SWANN

The generator has been built with the intention of producing a proton beam in excess of 5 million electron volts energy. This object has been attained.

The machine, whose total column height is 21 ft., was first built up to one-third of that height, successfully tested, and realized an analyzed beam of pure protons of about 50 microamperes at an energy of 2.0 Mev. The extension of the column to its full height yielded the aforementioned 5 million volts, and a proton beam of several hundred microamperes has been attained in focused form after passing through the analyzing magnet.

The energy of the beam has been measured to within three per cent by observing the magnetic field necessary to deviate it through a right angle, the magnetic field being obtained by use of a Pound magnetometer, with one value of the field standardized by the $Li(p,n)$ threshold.

BOOK REVIEWS

COMPLEX ANALYSIS, by Lars V. Ahlfors. 247 pages, diagrams, 16×24 cm. New York, McGraw-Hill Book Co., Inc., 1953. Price, $5.00.

Designed to serve graduate students and researchers alike, the International Series in Pure and Applied Mathematics has been in existence for a number of years. Its consulting editor, William Ted Martin, and the authors of earlier volumes have set and maintained high standards. Professor Ahlfors, if anything, has raised them further by giving his *Complex Analysis* the flavor of continental didactics. The book reflects at once the sublime and the genius of such great teachers as Bieberbach, Carathéodory, and Lindelöf. If the beginner sometimes wearies and thinks of mathematics as a dry subject, let him beware. Here he finds beauty and inspiration such as only the theory of analytic functions of complex variables has to offer. Held in suspense, he becomes a witness to the accomplishments of the classical theory as its modern interpretation unfolds before the avid reader.

Seven well-planned chapters treat related topics in form of subsections. The author prepares the reader firmly for the celebrated theorem of Cauchy and other advanced work. To that end, the first chapter on Complex Numbers reviews and revises notions which are commonly obtained on the undergraduate level "in a sloppy or insufficient manner" and goes on to modern algebra, the linear group, and the cross ratio. One gets an inkling of conformal mapping in connection with oriented circles. The second chapter deals with Complex Functions. The notion of continuity is followed by a preliminary definition of analytic functions, the Cauchy-Riemann differential equations, Laplace's equation, and harmonic functions. The author takes also time out to introduce important topological concepts needed to continue his theory. He obtains the lemma of Heine-Borel and the theorem of Bolzano-Weierstrass but he treats Jordan curves lightly as he employs them only occasionally. This chapter ends with conformal mapping and a thorough discussion of some elementary Riemann surfaces.

Now that a firm foundation has been laid, there follows that important chapter of Complex Integration. It begins with line integrals and the behavior of analytic functions on a rectangle or circular disc. Artin's winding number is used to discuss local properties of analytic functions. Taylor's theorem, local mappings, and Schwarz' lemma are obtained in rapid succession. Thus far, Cauchy's theorem has only been asserted for the circular region. Its generalization requires the extension of line integrals, the topology of simply and multiply connected regions, and the homology bases. The final formulation of Cauchy's theorem is due to Artin. Its consequence is the calculus of residues and principal values of integrals.

Infinite Sequences are introduced in the fourth chapter which includes factorization and infinite products. The gamma function leads to Stirling's formula but the exponential remainder term is left as an exercise. The next to last chapter takes the reader squarely into the twentieth century. Dirichlet's problem, the determination of a complex function from its boundary values, is presented in the formulation given by Oskar Perron. The Greens function is a natural sequel at this point. The final chapter deals briefly with multiple-valued functions, analytical continuation, and such topological questions as the homotopy group and the monodromy theorem.

The author has created "a text which is at once concise and rigorous, teachable and readable." Nowhere does he make any unrealistic or unrealizable demands of the student. His choice of topics and of deliberate omissions reveal discretion and teaching experience. The burden of preparing exercises is largely left to the teacher, as we find an average of four exercises per section, never more than nine. This maturity and the absence of "answers" may deter some teachers. There are a few errors of print, almost all very obvious.

The volume is dedicated to the memory of Ernst Lindelöf. Hardly could a better memorial have been erected to a more deserving mathematician and pedagogue. The fresh approach which is evident throughout the text will delight and inspire the students of graduate mathematics.

CARL HAMMER

RADIOISOTOPES IN INDUSTRY, edited by John R. Bradford. 309 pages, diagrams, illustrations, 16×24 cm. New York, Reinhold Publishing Corp., 1953. Price, $8.00.

In view of the widespread use of the word "isotope" it is surprising how few people understand the tremendous potential in industrial progress the word represents. It is hoped that this book, based on a conference at Case Institute of Technology in April, 1951, will prove to be the initial step toward a remediation of this situation.

Considering first the technical aspects of the book, one finds them generally quite good. The index is adequate which is unusual in books of this type, and the illustrations are clear, simple and helpful. The book contains a minimum of mathematics and requires no special training in chemistry or physics in order to gain a complete understanding of the material. There is, perhaps, a little too much repetition, but in a collection of conference talks this seems unavoidable. Finally, one could have hoped for a chapter stating more strongly and more specifically the dangers in working with radioactive material. A few of the "horror" stories so widely circulated among health physicists might have helped to impress those workers one always encounters, whose care and judgment in using a new tool are proportional to the fear the tool instills in them.

Apart from the technical aspects this book promises to do a great service for industry as a whole, and the isotope industry in particular. More important than the many concrete suggestions as to uses of isotopes in industry, is the fact that most suggestions are made in generalized terms which give one the impression that "here is something I can use in my work."

Radioisotopes in Industry also presents, in the first chapter, an eloquent case for the government operation of the industry, at least for the present. That government participation has been good for this field is undeniable. Also in this book, one hears a plea, largely between the lines, for more health physicists and a more efficient use of the ones available. This plea must be heard before the industry can assume its rightful proportions.

This book can be heartily recommended for reading by present and potential isotope users in industry, as well as students who hope to enter progressive industries. The research worker in fundamental science will not find many answers to his basic questions, but he will find the book interesting reading, and the chapters on laboratory designs and radiation hazards very helpful.

All in all, this book deserves an enthusiastic reception in a field which increased its membership four-fold within three years after its principle tool became easily available.

FRANK E. JAUMOT, JR.

PHYSICAL CHEMISTRY OF METALS, by Lawrence S. Darken and Robert W. Gurry. 535 pages, illustrations, diagrams, 16×24 cm. New York, McGraw-Hill Book Co., Inc., 1953. Price, $8.50.

Although excellent texts on physical chemistry and thermodynamics are available, they are useful to the metallurgical student more as reference books than as textbooks. The present book is designed especially for the metallurgist and fills a long standing vacancy in the well known Metallurgy and Metallurgical Engineering Series. The authors have been two of the leading exponents for the application of the thermodynamic method to metallurgical problems. Drs. Darken and Gurry have contributed many important papers on the physical chemistry of metals and again demonstrate their discerning ability by performing a notable job in filtering the vast field of physical chemistry and thermodynamics in order to delineate the fundamental principles which are of special concern to the student of metals.

The first five chapters of the book deal with the gaseous, liquid, and solid states, with a chapter on solid solutions and intermetallic compounds. At all times emphasis is placed on the chemistry of the metallic state. The authors' presentation of the solid state is particularly good. In this chapter they discuss the periodic table, atomic structure, crystal bonding, plastic deformation, Hume-Rothery's classification of the elements, atomic radius of the elements, Pauling's theory of atomic radius for metals, and imperfections in crystals. It seems impossible that these important subjects could be covered in fifty pages, yet the authors do a creditable job. A short discussion of plastic flow in polycrystalline metals and work hardening

involving a perfunctory explanation in terms of dislocation, detracts from the continuity of this chapter since it preceeds the section on imperfections in crystals.

Chapters six, seven, and eight are devoted to the classical treatment of the first, second, and third laws of thermodynamics. Chapters nine and ten discuss fugacity, activity, the equilibrium constant, and solutions. The chapter on solutions is very well written and in itself justifies the use of this book as a reference. The authors discuss in detail such subjects as the determination of the activities of one component when that of the other is known and the conversion from one standard state to another. Much of the material in this chapter has not appeared in textbooks before. Chapters eleven, twelve, and thirteen deal with phase relations, heterogeneous equilibria, and free-energy-composition diagrams. These subjects have been presented in many texts but seldom with such clarity and detail as is shown here. This section provides the information necessary for a thorough understanding of the phase diagram, an understanding that is mandatory for the metallurgist. Chapter fourteen is worthy of special mention since it concerns the important but often neglected topic of temperature-pressure diagrams. The relation between temperature and pressure for a univariant equilibrium in a binary system is developed. Equilibria in the following systems are discussed: metal-oxide, metal-sulfide, metal-carbide, metal-nitride and metal-chloride. Chapters fifteen and sixteen involve the thermodynamics of the iron-nitrogen and iron-carbon systems. The authors state that they believe these chapters to be more comprehensive than can be found in the present literature. Perusal of the material in this section of the book reveals little reason to question their statement. Although these two systems serve to illustrate certain thermodynamic principles under discussion, it is believed that they have been allocated too much space in a textbook of this type.

Electrochemistry as concerned with reversible cells is the subject of chapter seventeen. The authors mention that such cells offer an important means of determining the activities of components in solutions. The student usually has difficulty in attaining a "feeling" for activities. One of the best means of assisting him to overcome this difficulty is to demonstrate how activities are actually obtained by electromotive force measurements. In this chapter such an illustration in detail would have fit in quite well with the material on activities as discussed in chapter ten. The final two chapters serve as introductions to the theory of diffusion and the kinetics of metallurgical processes. These are reproductions of two of Dr. Darken's previous papers appearing in the books *Atom Movements* (American Society for Metals, Cleveland, 1951) and *Basic Open Hearth Steelmaking* (American Institute of Mining and Metallurgical Engineers, New York, 1951). Finally, the book is supplemented by an extensive collection of problems by Dr. M. B. Bever.

This is an outstanding text, very well written, adequate in coverage, and containing much new material. It could be used to great advantage as a textbook for the senior or graduate metallurgist in courses dealing with the physical chemistry of metals.

RAYMOND L. SMITH

THE COMETS AND THEIR ORIGIN, by R. A. Lyttleton. 173 pages, diagrams, plates, 15×23 cm. New York, Cambridge University Press, 1953. Price, $5.00.

The most mysterious, fascinating and spectacular phenomenon in the sky is the bright comet. For thousands of years man has observed and studied these transient objects without learning the answer to the provocative question, "How did they originate?" They appear so infrequently and their physical characteristics are such that astronomers cannot get a hold on them.

As a result there is an almost complete lack of understanding of the composition, behavior and origin of comets. Within the past few years one theory has been proposed by Dr. F. W. Whipple of the Harvard College Observatory. However, many questions remain unanswered and the origin of comets is still highly speculative.

As a possible answer to these questions there appears on the astronomical scene the book, *The Comets and Their Origin* by Dr. R. A. Lyttleton, fellow of St. John's College, Cambridge. Dr. Lyttleton is of a new school of theoretical astronomers headed by the articulate Fred Hoyle.

Within the new school many subjects relating to stellar evolution have been debated and solutions sought for pressing astrophysical problems. This book is the outgrowth of this approach.

Dr. Lyttleton believes that comets originated by the sun sweeping through clouds of galactic dust. He shows that as a result of this passage the clouds will develop into aggregations of dust particles describing parabolic orbits around the sun. The mechanism for this is the collisions which take place between particles. He shows that collisions can reduce the originally hyperbolic energies of the particles to elliptic values and thus bring about their capture by the sun. This is the crucial point of the book. Herein is contained the mechanism which can create comets from interstellar dust clouds and provide the means to effect capture by the sun.

He shows that the process may be taking place today, if the sun were passing through a dust cloud. However, were this the case we would be unaware of the formation of comets. The original dust cloud may be too tenuous to involve obscuration of the nearer stars and the formation process takes place at distances which are too great for the birth to be seen.

The idea of cometary formation in this fashion was first hinted at by Princeton's Dr. H. N. Russell in his challenging discussion of the solar system and its origin. This appears to be the starting point for Dr. Lyttleton who develops his approach with a keen insight and a new comprehension of the problem. For the first time there is presented a hypothesis which can be subject to quantative analysis—a feature which previously has never met with success. Also, in all other theories the origin of the planets and comets were considered coeval. Here we find the two represent fundamentally different classes of celestial objects; one being formed 3 or 4 billions of years ago, the other perhaps in the process of formation today.

The book is a brilliant piece of exposition and while it is highly speculative and will be vigorously debated, the book presents a fresh, ingenious approach which points the way to a possible solution of the problem of the origin of the comets. I. M. LEVITT

EVALUATING RESEARCH AND DEVELOPMENT, edited by Irving R. Weschler and Paula Brown. 104 pages, 15×23 cm. Los Angeles, University of California (Institute of Industrial Relations), 1953. Price, $1.65 (paper).

Within the last hundred years we have seen the decline in importance of owner-managed industries and the simultaneous growth of huge industrial complexes governed by specialists in management. Starting with the formation of Edison's laboratory and continuing at an accelerated rate during the last twenty years, huge industrial, academic, foundation, institutional and government scientific research organizations have developed in this country. The administration and direction of these research activities has required the development of a specialized managerial group in much the same way as has expanding industry. The book under review is essentially an annotated proceedings of a conference of local research administrators held at UCLA. The two major problems under discussion were (1) the formulation and definition of research and development objectives, and (2) the evaluation of individual and group performance with reference to objectives.

Although it is of interest to read the sifted comments of the distinguished conferees, this book makes it quite clear, perhaps unintentionally, that precious little is known about the goals and techniques of research management. The discussion of research objectives was rather trite. The discussions on evaluation were of interest in that they pointed up a central problem in research management. The participants agreed that in order to evaluate a piece of technical work the evaluator must be able to perform the work himself; that is, an evaluation by a man's technical peers is implied. This situation leads to obvious difficulties above the project engineer level in a large organization.

The book will not be helpful to research directors because of the lack of any clearly formulated principles on which to base action. This failing is a general characteristic of present study programs in the field of research management. What is needed is a bold new departure, rather than the present strict adherence to classical industrial psychology procedures.

EZRA KRENDEL

SMALL PARTICLE STATISTICS, by G. Herdan (With a Guide to the Experimental Design of Particle Size Determinations by M. L. Smith). 520 pages, tables, 15×23 cm. Houston, Elsevier Press, 1953. Price, $12.00.

The statistics of numbers and statistical physics both have been extensively investigated and reported on in recent years. The statistics of particles in the range of several centimeters to approximately 10^{-5} centimeters, on the other hand, has been relatively neglected. It is the author's purpose to establish the theory and practice of small particle statistics with its special problems and application as a branch of statistics in its own right.

The book assumes no intimate knowledge of statistics and all concepts are developed as they are needed. Of especial interest and value is the table of symbols with their meanings that are used in the book.

The first part of this book is concerned with the building up of the statistical tools necessary for small particle size analysis and the representation of their sizes and distribution. Of especial value is one chapter dealing with the differences in determination results. These differences may arise from different methods of determination, different workers doing the determination, differences in sampling technique, etc. The author shows how these differences may be correlated, tested and combined.

Part II deals with the relation of the particle diameter to its other physical characteristics, such as surface area, shape, porosity, etc. In many technological applications it is one of the above, and not the size alone, that determines the physical and chemical characteristics of the particle. For example, the surface area of a catalyst is one of the determining factors in its action. Thus in Part II the statistics for the dependence of these factors on each other are developed. In later chapters specific examples are shown.

In Part III the author discusses how particle size distribution may be decreased by grinding and crushing or increased by flocculation or demulsification. The changes in distribution that occur in these processes are investigated. A chapter on the inhomogeneity of polymers of high molecular weight is included.

Part IV of this book shows the relationship of small particle statistics to the number of statistics on the higher side and to the statistical physics on the lower side. Both the theoretical and experimental aspects of this problem are discussed.

The last section of the book, written by Dr. M. L. Smith, is concerned with the experimental methods for measuring particle size, shape, surface area, etc. Among the methods described in more detail are microscopic determination, various sedimentation methods, the air permeability method and the adsorption of gases at low temperatures. Limitations and advantages of these various methods are discussed in detail.

This book fulfills the need of workers in the industries and laboratories in which the particle size of materials is an important factor. It is at the same time a handbook and text book. Its systematic arrangement provides easily accessible information, while for the novice in small particle work thorough study would be of benefit. Throughout the book many numerical examples are given which aid greatly in the understanding of the material. There are numerous up-to-date references at the end of each chapter. These references together with the detail in the chapter itself form a very complete study of the statistics of small particles.

ROBERT B. CAMPBELL

VACUUM TUBE OSCILLATORS, by William A. Edson. 476 pages, diagrams, 16×24 cm. New York, John Wiley & Sons, Inc., 1953. Price, $7.50.

Although vacuum tube oscillators have been the subject of continuous research and development, reported in many technical papers, there have been few general books on the subject in recent years. Professor Edson has undertaken to bring together in a coherent presentation the large amount of material hitherto available only in scattered journals.

The book has been planned as a classroom text for senior or graduate electrical engineering students, and the material has been so used by Professor Edson at Georgia Institute of Technology. However, this reviewer feels that the book will find wider use as a reference for practicing electronic engineers, or for students in less specialized courses in vacuum tube circuits.

The treatment is somewhat uneven in its level. In his preface, the author states that he has favored ". . . the viewpoint of design over that of analysis because it represents the basic purpose of engineering and because the ability to design is a-priori proof of competence in analysis." This would seem to indicate that a book which hopes to teach design should establish a firm basis in analysis. Just so, the best chapters in the book are those which are analytic in character.

The first five chapters fit this description, with those on "Transient Behavior of Linear Systems" and "Feedback Systems and Stability Criteria" being particularly good as brief useful statements of their subject matter. Chapter 6, "Resonators" descends to a handbook level and lacks lucidity in places. For example, consider the statement, concerning piezoelectric crystals, "It may be helpful to note that the AT and CT cuts are nearly parallel to one of the three pyramidal caps of a perfect natural crystal." Now, to a student who remembers the "perfect natural" crystals in the museum as hexagonal prisms capped by *two* pyramids, and to whom the idea of parallelism between dissimilar solid bodies is obscure, this doesn't help at all.

The following chapters on linear oscillators, harmonic oscillators and crystal controlled oscillators contain many design data and brief analytic sections. Chapters 10 and 11 are concerned with operating characteristics, the first giving an extension of the Nyquist criterion. Chapter 12 is a compact discussion of free-running relaxation oscillators. Since the book omits magnetrons and klystrons on the plea that there is adequate modern material available, it seems that this chapter could also have been omitted for the same reason.

Chapters 13 and 14 deal with application problems such as synchronization and frequency changing. Chapter 15 on "Tube and Thermal Noise" could be expanded for clarity. At the bottom of page 373 the word "less" seems to have crept in where "greater" should be, in the next to the last sentence. Read literally, the conclusion is reached that for the case at hand the probability that a random noise peak exceeds 3.4 times the rms value is unity!

The book closes with three chapters on modulation, automatic frequency control and long-line effects. The bibliography is extensive, containing 352 entries. It would be more useful as a reference tool if it had been arranged in sections according to special subjects, rather than alphabetically by author.

A. C. BYERS

ADVANCES IN GEOPHYSICS, Vol. I, edited by H. E. Landsberg. 362 pages, maps, diagrams, 16×24 cm. New York, Academic Press, Inc., 1952. Price, $7.80.

The tremendous surge in geophysical research since the early 1940's has posed a serious problem to the professional and advanced student because of the inability to keep abreast of the latest development in this exciting field. The problem has been further complicated by over-specialization and the tendency to restrict efforts to unusually limited phases. This trend has retarded the integration of the geophysical field into a revealing picture for the student or reader with limited research time. As a result undue time and effort must be expended to keep ahead of the field.

Indicative of this trend is the enormous volume of reference material found at the conclusion of each article and consulted in the preparation of these papers. By actual count 450 papers are cited!

Now comes a welcome addition to the literature in the form of Volume I of a geophysical series. *Advances in Geophysics* is an attempt, in a series of monographs, to summarize advances in geophysics and show a picture of the field as viewed today by the leaders in this field. It will prove a tremendous aid to specialists in that these monographs will comprise a series of survey articles to keep them abreast of current trends.

Dr. H. E. Landsberg, of the Cambridge Air Force Research Center, has shrewdly selected eight significant topics to be incorporated in this first volume. The first two are unique, authoritative and long overdue. As the editor says in his foreword, "Two of the papers deal with problems of evaluating geophysical data. The rate at which such data have accumulated is staggering. This concerns all fields of geophysics; they are all plagued by vast archives of raw material. Evaluation and interpretation is imperative and the old-fashioned approaches are manifestly inadequate."

The other six deal with the general circulation of the atmosphere; the ocean action in estuaries; the gravity field of the earth; airborne magnetic survey methods; characteristics of the atmosphere as determined by meteors and finally the unsolved problems in physics of the upper atmosphere.

Dr. Landsberg is to be congratulated in his choice of subject matter to comprise the first volume. It is sincerely hoped that the other volumes will match the first in authority, scope and interest.

I. M. LEVITT

INDUSTRIAL ELECTRONIC ENGINEERING, by Wells L. Davis and Herman R. Weed. 514 pages, diagrams, tables, 14×22 cm. New York, Prentice-Hall, Inc., 1953. Price, $8.50.

Good books on the engineering aspects of electronics are presently in great demand. This is particularly true since there are not enough trained and experienced electronic engineers to go around. Certainly a well-written textbook can convey a useful amount of information about the principles of operation and the prevalent design practices associated with electronic equipment. This type of book in the hands of an intelligent individual with technical aptitudes provides a combination adequate to many of our engineering needs. More elaborate preparation than this is, for many jobs, a waste of time.

Furthermore, experience in the field of industrial electronics has shown that this fascinating subject attracts the interest of many technical specialists from other branches of science. This is logical for many reasons, but the fact that these people need ready information on certain popular electronic devices such as servos and control systems is the significant point. The list of these industrially important electronic devices corresponds very nearly to the Contents of Davis' and Weed's book: Gas Tube Circuits, Rectifiers and Inverters, Phototubes, Timers, Servomechanisms, Electronic Control of Motors, Radio Frequency Heating, X-Rays (and radiation applications).

Davis and Weed have apparently distilled from more comprehensive sources the ingredients for their chapters. This has been done with some skill, for in studying this material one obtains a little of the theoretical basis for each subject as well as the key facts of design which predict circuit performance. This is not one of the so-called "simplified" books; rather, it is an abbreviated treatment, definitely directed at industrial electronics. No mention is made of the large but unrelated branches of electronics such as television, microwaves or general communications. Exemplary circuits have been selected to illustrate current practice in the categories mentioned above, and the corresponding rigorous mathematical expressions are developed for each case. Tables, curves and diagrammatic sketches also serve to clarify the text.

Problems of a type that call for the derivation of mathematical expressions to satisfy stated conditions are provided as exercises and illustrations. For teaching purposes this volume can be used in an elective course perhaps better than for the instruction of students majoring in electronics. The amount of specialization required in the latter case is greater than this book affords.

C. W. HARGENS

DIFFERENTIAL AND INTEGRAL CALCULUS, by Philip Franklin. 641 pages, diagrams, 16×24 cm. New York, McGraw-Hill Book Co., Inc., 1953. Price, $6.00.

Professor Philip Franklin was eminently qualified to write this treatise on calculus. He has to his credit two texts on the advanced level and a book on Fourier methods. His *Differential and Integral Calculus* contains enough material for a two or three semester course but it deviates little from other, almost standardized texts on the same subject. Some of its material is modeled after the author's more advanced texts and the whole volume is written in the lucid style which characterizes all of his works.

Professor Franklin follows the modern trend and introduces integration of power functions very soon after he is on terra firma with differentiation. He returns to the integration of other functions some 178 pages later. The discussion of simple and damped harmonic motion leads quite early to the solution of linear, homogeneous differential equations which are treated more

fully in the last chapter. Other chapters present a review of trigonometric, exponential, and logarithmic functions, or the highlights of plane and solid analytical geometry. A chapter on vectors, similar to that in the author's advanced text, lends weight to the modern approach on the undergraduate level. Many facets of the calculus are discussed in greater detail than usual; relations between derivatives of parametric functions, force components in polar coordinates, Duhamel's theories, and extrema of functions of two or more variables are examples of such refinements.

Numerous exercises are worked out in detail after each chapter. They are set in small type which is sometimes difficult to read. Besides the usual calculus problems we find some novel ones and also examples from all branches of physics. This accentuates the absence of probability theory and mathematical statistics which assume gradually increasing importance. Hyperbolic functions are treated very lightly, but the author has generally lived up to his promise to write a book which is precise and suitable for engineers and science students. The publishers have spared no effort to make this an attractive volume and kept the price at a nominal level. It seems quite likely that Professor Franklin's newest text will make a dent into the market, as it rightly should. CARL HAMMER

METADYNE STATICS, by Joseph Maximus Pestarini. 415 pages, illustrations, diagrams, 16×24 cm. Cambridge, The Technology Press; New York, John Wiley & Sons, Inc.; 1952. Price, $9.00.

About thirty years ago, Dr. Pestarini investigated some problems in commutation of rotating direct current machines. Out of that work he developed the design procedures and supervised the construction of many of a group of rotating machines called metadynes. That he is the world's leading authority in this field which he created is attested to by the fact that he is the holder of 256 patents on the subject.

A metadyne is a rotating machine consisting basically of an armature (similar to any of the conventional varieties used in direct current machines) and a commutator, and provided with at least three brushes per cycle of the machine structure. Commutation is independent of the windings on a stator or of any magnetic member acting on the armature. Such a machine can be designed to have the electrical characteristics of a transformer, a generator, a motor, or a combination of those characteristics simultaneously.

This book, devoted to the basic concepts of the static or steady state operating characteristics of metadynes, is the first of a series of volumes on the subject prepared by Dr. Pestarini. The second volume, the publication of which is promised for the near future, will treat of the dynamic or transient characteristics which exist when the metadyne is subjected to rapidly varying currents.

The first part of the book is an exposition of the general theory of metadyne design and operation. The relationships between the various currents and voltages are developed, starting with the simplest metadyne consisting of an armature and commutator and continuing on through metadynes with arbitrary arrangements of brushes and stator windings. The vast number of design variables allows the construction of machines having almost an endless variety of operating characteristics. Many of these are discussed, including those designs made for operation at constant speed, approximately constant speed, and substantially variable speed according to some arbitrary relation.

The second part of the book is devoted to applications of metadynes to constant voltage or constant current systems and as converters from one of these systems to the other. Following a chapter devoted to the detailed consideration of the crossed transformer metadyne (wherein for a four-brush machine, non-consecutive pairs of brushes are connected to some device), numerous examples of special designs are worked out. One of these is the amplifying metadyne which in the United States is widely sold under the name of amplidyne.

Metadynes are most efficiently used when their characteristics can be chosen as part of the synthesis of an operating system of machines. This being the case, they perhaps are not fully suitable for mass production methods, which may be one reason for their lack of widespread acceptance. If this book, however, is read in detail and the potentialities of these

devices fully understood, then we may see wider application of the metadyne. It is recommended for all serious students of rotating machinery and for designers of electrical power drives.

S. CHARP

MESON PHYSICS, by Robert E. Marshak. 378 pages, diagrams, 16×24 cm. New York, McGraw-Hill Book Co., Inc., 1952. Price, $7.50.

The μ meson was discovered around 1936, the π meson was discovered in 1947 and the π meson was produced artificially in 1948. The extent of meson research since 1947 is obvious from the fact that an entire book can be devoted to the theoretical aspects of mesons. *Meson Physics* is intended for the advanced research worker and not for a beginner. The prerequisites are given as quantum mechanics, quantum theory of radiation, and quantum theory of fields. It is rather surprising that there is no introduction in the form of a brief statement as to what a meson is, how it was first discovered, and what its place in nuclear physics is. It is assumed that the reader is not a tyro.

This is mainly a book about π mesons; six and a half chapters of the nine in text are devoted to them. One and a half chapters are given over to the properties of the older μ meson, and a final chapter is devoted to the newer elementary particles and to new processes. Of the chapters on π mesons, four are given over to means of production, one to properties, one to nuclear reactions and one to absorption.

With a proper theoretical background, the researcher should find this book highly readable. The style is clear and informal and the material is presented in lucid fashion. The author discusses meson properties and uses these to make critical evaluations of theory. Throughout the text there is constant comparison of theory and experiment. The author omitted a discussion of the role of meson theory in nuclear forces since he regards this subject as too spectacular. Interestingly enough, the Yukawa theory of nuclear exchange forces came before the actual discovery of mesons.

While much of the material in this book is still in the uncertain stage, Professor Marshak finds that the theory of real meson processes possesses "a qualitative content which is in surprising accord with experiment."

LEONARD MULDAWER

BOOK NOTES

SEWERAGE AND SEWAGE TREATMENT, by Harold E. Babbitt. Seventh edition, 674 pages, illustrations, diagrams, 16×24 cm. New York, John Wiley & Sons, Inc., 1953. Price, $8.00.

The 48 new features of the seventh edition of this indispensable treatise include such items as the latest sewerage census statistics, the new standard design specifications published by various State Boards of Health, pumping station design data, the hygienic aspects of sludge utilization, a section on detergents, descriptions of several new processes (Hay's, Zigerli, Kraus, Okun and Fair), new data on scum control, and data on the disposal of radioactive wastes. For those who have depended on prior editions, this new version is a "must."

PHYSICAL CHEMISTRY FOR COLLEGES, by E. B. Millard. Seventh edition, 618 pages, diagrams, 16×24 cm. New York, McGraw-Hill Book Co., Inc., 1953. Price, $6.00.

Representing almost a complete revision as far as text matter and illustrative material are concerned, this new seventh edition of a standard text nevertheless follows the main pattern of the sixth edition (1946). New problems have been added, and the longer lists of problems have been divided into two groups, according to difficulty, with the easier problems listed first. The one change in chapter order places the thermochemistry chapter immediately following the one on thermodynamics. In its up-to-date version, this text will continue to stimulate students.

CURRENT TOPICS

Multiplex Radio Transmission.—A system of multiplex radio transmission has been perfected by Dr. Edwin H. Armstrong, professor of Electrical Engineering at Columbia University, and John H. Bose, of Columbia's Marcellus Hartley Research Laboratories. This system, which enables FM broadcasting stations to transmit simultaneously two or more different programs, is the most recent development based on principles which were described by Dr. Armstrong in 1935. The new development, however, employs a number of improvements created since World War II. The result, according to Dr. Armstrong, will be to double, at least, the effective function of the FM transmitter, "with obvious far-reaching effects on the methods of broadcasting radio communications of all kinds."

Several new types of service are made possible by this new system. An FM station, for example, can transmit a musical program on one channel, and send out on a second channel a voice program such as a speech, news bulletins or an advertising program; or it can transmit a single program stereophonically on the two channels. It can also act to transmit two independent musical programs so as to relay along a program to another station which it itself is not carrying on the main channel. All of these methods of operation have been successfully tested at Dr. Armstrong's Alpine FM transmitter, KE2XCC, at Alpine, N. J. Dr. Armstrong added that, of course, the system would not be limited to two channels only.

The present-day FM set is quite unaffected by the presence of the second channel, Dr. Armstrong said. Listeners on present-day FM sets, in fact, are unaware that there is a second program from Alpine that they are missing. Such two-channel transmissions have passed unnoticed for several years from the Alpine station. In order to receive the second channel, a modified form of FM receiver will be necessary, but the number of additional parts required will not unduly increase the cost of the set.

While multiplex operation of FM in the communications field has been in practical use for some years, its use in broadcasting presented problems not important in the communications field. These problems have been studied and successfully disposed of in several years of quiet work in the Marcellus Hartley Research Laboratories at Columbia and in experimental tests at the Alpine transmitter, with the result that the system has been brought to a practical entity, according to Dr. Armstrong. The credit for the engineering of the system, he said, belongs to Mr. Rose who, with Perry Osborn, chief engineer of the Alpine station, directed the experimental work.

He added that the real significance of the new system is that FM will prevail as the final aural system. It would be completely impossible to multiplex on any AM system, either on the standard band, or on a very high frequency (v.h.f.) band of the same frequency as the FM stations, and a station operator who can sell time on two channels obviously will outsell eventually the station operator with only one. The cost of equipping a transmitter for multiplex operation is not high. Transmitters already equipped with Serrasoid modulators which are employed in the new system are most readily susceptible to change, he said. Receiver prices will doubtless follow the course they have always taken since the superheterodyne receiver was introduced, producing a set ultimately fitting the capabilities of everyone's pocket-book.

INDEX TO VOLUME 255

January to June, 1953

References with (N) are Notes from the National Bureau of Standards, those with (F) are Notes from The Franklin Institute Laboratories, and those with (C) are Current Topics.

A

Aerial mapping. Wide-angle aerial lens, 169(C)
Air conditioning. All-electric heat pump, 332(C)
Airplanes. Industry provides own fire protection, 138(C)
Alphatron vacuum gauge, 364(C)
Aluminum. Aluminum bridge for Army, 169(C)
  Lightweight triangulation tower, 288(C)
Ammunition. Army reclaims old ammunition, 8(C)
  High voltage fluorography, 352(C)
Analysis. Nonlinear analysis of electro-mechanical problems (Ku), 9
Analytical mechanics. Powerless glides along zero curvature flight paths (Bolie), 127
Analyzers. An a-c network analyzer, 344(F)
Antioxidants. Conidendrin-derived compounds tested as stabilizers, 496(C)
Applied mathematics. Asymptotic expansion of the Whittaker's function $W_{k,m}(z)$ for large values of k, m, z (Chang, Chu and O'Brien), 215, 319
  Powerless glides along zero curvature flight paths (Bolie), 127
Approach light systems. The kinorama—a device for evaluating airport approach lights, 237(N)
Aqualung, 288(C)
Armor. Marine Corps body armor, 267(C)
Arsove, Maynard G.: The algebraic theory of linear transmission networks, 301, 427
Artillery. The engineering development of the 280-mm. weapon, 565(F)
Asymptotic expansion of the Whittaker's function $W_{k,m}(z)$ for large values of k, m, z (Chang, Chu and O'Brien), 215, 319
Atmospheric transmission. The infrared transmission of atmospheric windows (Elder and Strong), 189
Atomic energy. Nuclear power and industry (Grebe), 409
Atomic gun. The engineering development of the 280-mm. weapon, 565(F)
Atomic physics. AEC survey shows growth of radiation instrument industry, 363(C)
  "Hot lab"—atomic age tool, 167(C)
  Radioactive waste gases, 165(C)
Autopilot. High-speed "brains" for high-speed planes, 166(C)
Aviation. An airborne, infrared spectrograph, 59(F)
  Engineering evaluation of flight testing dive or toss bomb computers, 567(F)
  Hydro-ski landing gears, 168(C)
  The kinorama—a device for evaluating airport approach lights, 237(N)
  One-man helicopter, 473(C)
  Powerless glides along zero curvature flight paths (Bolie), 127
  "Supersonic" eyes, 352(C)
Axles. New axle tester, 300(C)

B

Blowers. A Diesel engine blower test stand, 8(C)
Body armor. Marine Corps body armor, 267(C)
Bolie, Victor W.: Powerless glides along zero curvature flight paths, 127
Book reviews
  Abetti, Giorgio: The history of astronomy, 156
  Ahlfors, Lars V.: Complex analysis, 572
  Albert, Arthur L.: Electrical fundamentals of communication, 85
  Aldous, W. H. and Sir Edward Appleton: Thermionic vacuum tubes and their applications, 468

Anderson, R. L. and T. A. Bancroft: Statistical theory in research, 358
Annual review of nuclear science, Vol. 2, 463
Babbitt, Harold E.: Sewerage and sewage treatment, 580
Boltz, David F., ed.: Selected topics in modern instrumental analysis, 465
Bond, Horatio: A "first book" on fire safety in the atomic age, 85
Bosworth, R. C. L.: Heat transfer phenomena, 465
Böttcher, C. J. F.: Theory of electric polarisation, 359
Boundy, R. H. and R. F. Boyer: Styrene, its polymers, copolymers and derivatives, 78
Bowden, F. P. and A. D. Yoffe: Initiation and growth of explosions in liquids and solids, 155
Bradford, John R., ed.: Radiosotopes in industry, 573
Brescia, Frank: General college chemistry, 468
Bronwell, Arthur: Advanced mathematics in physics and engineering, 464
Chapman, Sydney and T. G. Cowling: The mathematical theory of non-uniform gases, 361
Charnes, A., W. W. Cooper and A. Henderson: An introduction to linear programming, 467
Code for protection against lightning, 361
Conway, B. E.: Electrochemical data, 82
Curme, George O., Jr. and Franklin Johnston, ed.: Glycols, 84
Darken, Lawrence S. and Robert W. Gurry: Physical chemistry of metals, 573
Davis, Wells L. and Herman R. Weed: Industrial electronic engineering, 578
Diehl, Harvey and G. Frederick Smith: Quantitative analysis, 255
Doan, Gilbert E.: The principles of physical metallurgy, 255
Dow, William G.: Fundamentals of engineering electronics, 85
Drysdale, C. V. and A. C. Jolley: Electrical measuring instruments, 159
Edson, William A.: Vacuum tube oscillators, 576
Efficient pickling with "Rodine," 255
Faires, Virgil Moring and Sherman Daniel Chambers: Analytic mechanics, 256
Fairman, Seibert and Chester S. Cutshall: Mechanics of materials, 464
Farber, Eduard: The evolution of chemistry, 80
Farrall, Arthur W.: Dairy engineering, 361
Feather, N.: Nuclear stability rules, 353
Ferguson, Lloyd N.: Electron structures of organic molecules, 249
Franklin, Philip: Differential and integral calculus, 578
Frost, A. A. and R. G. Pearson: Kinetics and mechanism: a study of homogeneous chemical reactions, 355
Furry, W. H., E. M. Purcell and J. C. Street: Physics for science and engineering students, 256
Gatland, Kenneth W.: Development of the guided missile, 253
Goulden, Cyril H.: Methods of statistical analysis, 86
Grew, K. E. and T. L. Ibbs: Thermal diffusion in gases, 154
Hall, A. R.: Ballistics in the Seventeenth Century, 81
Hall, W. J. and others, ed.: Review of textile progress, 361
Harwell: The British atomic energy research establishment, 160
Harwood, Paisley B.: Control of electric motors, 158
Herdan, G.: Small particle statistics, 576
Hildebrand, F. B.: Methods of applied mathematics, 460
Hitchins, H. L. and W. E. May: From lodestone to gyrocompass, 461
Hoyt, S. L.: Metal data, 255
Hurd, Dallas T.: An introduction to the chemistry of hydrides, 77
Jacobson, C. A., ed.: Encyclopedia of chemical reactions, Vol. V, 463
Jaynes, E. T.: Ferroelectricity, 460
Jones, George A.: High speed photography, 461
Knoll, M. and B. Kazan: Storage tubes and their basic principles, 84
Kolthoff, I. M. and E. B. Sandell: Textbook of quantitative inorganic analysis, 85
Landsberg, H. E., ed.: Advances in geophysics, Vol. I, 577
Liddicoat, R. T. and Philip O. Potts: Laboratory manual of materials testing, 468

Little, Arthur and K. A. Mitchell: Tablet making, 158
Lyttleton, R. A.: The comets and their origin, 574
MacDougall, Frank H.: Physical chemistry, 160
Marine fouling and its prevention, 158
Marshak, Robert E.: Meson physics, 580
Martell, Arthur E. and Melvin Calvin: Chemistry of the metal chelate compounds, 250
McKie, Douglas: Antoine Lavoisier, 77
Menzel, Donald H.: Flying saucers, 360
Millard, E. B.: Physical chemistry for colleges, 580
Milne, E. A.: Sir James Jeans, 358
Moeller, Therald: Inorganic chemistry, 153
Mole, John Henry: Filter design data for communication engineers, 249
Montagu, Ashley: Darwin: competition and cooperation, 251
Moss, T. S.: Photoconductivity in the elements, 254
Muchmore, Robert B.: Essentials of microwaves, 157
Murray, W. M., ed.: Fatigue and fracture of metals, 155
Nord, Melvin: Textbook of engineering materials, 357
Parker, Milton E., E. S. Harvey and E. S. Stateler: Elements of food engineering, 255
Payne-Gaposchkin, Cecilia: Stars in the making, 80
Peck, Ralph B., Walter E. Hanson and Thomas H. Thornburn: Foundation engineering, 459
Pepinsky, Ray, ed.: Computing methods and the phase problem in x-ray crystal analysis, 356
Perlis, Sam: Theory of matrices, 83
Pestarini, Joseph Maximus: Metadyne statics, 579
Porter, J. G.: Comets and meteor streams, 251
Reintjes, J. Francis and Godfrey T. Coate: Principles of radar, 85
Review of electronic digital computers, 82
Richards, Horace G.: Record of the rocks, 355
Rochester, G. D. and J. G. Wilson: Cloud chamber photographs of the cosmic radiation, 76
Rossi, Bruno: High-energy particles, 78
Ryder, John D.: Electronic engineering principles, 256
Schrodinger, Erwin: Statistical thermodynamics, 361
Sechler, Ernest E.: Elasticity in engineering, 252
Seely, Fred B. and James O. Smith: Advanced mechanics of materials, 159
Sherman, Henry C.: Chemistry of foods and nutrition, 158
Shoenberg, D.: Superconductivity, 253
Sommer, A.: Photoelectric tubes, 468
Sommerfeld, Arnold: Electrodynamics, 81
Sommerfeld, Arnold: Mechanics. Lectures on theoretical physics, Vol. 1, 354
Speel, Henry C., ed.: Textile chemicals and auxiliaries, 252
Standing, M. B.: Volumetric and phase behavior of oil field hydrocarbon systems, 353
Staniland, L. N.: The principles of line illustration, 465
Stieri, Emanuele: Basic welding principles, 467
Sutton, O. G.: Micrometeorology, 356
Thaler, George J. and Robert G. Brown: Servomechanism analysis, 462
Thorndike, Alan M.: Mesons—a summary of experimental facts, 458
Tippett, L. H. C.: The methods of statistics, 159
Trorey, Lyle G.: Aerial mapping and photogrammetry, 86
Tustin, A.: Direct current machines for control systems, 153
Vail, James G.: Soluble silicates, Vol. 2, 254
Venkataraman, K.: The chemistry of synthetic dyes, Vol. II, 466
Von Tersch, L. W. and A. W. Swago: Recurrent electrical transients, 466
Weschler, Irving R. and Paula Brown, ed.: Evaluating research and development, 575
Westergaard, H. M.: Theory of elasticity and plasticity, 157
Wilson, E. Bright, Jr.: An introduction to scientific research, 357
Winans, John Gibson: Introductory general physics, 159
Wise, Louis E. and Edwin C. Jahn, ed.: Wood chemistry, Vol. 1, 86; Vol. 2, 159

Wood, R. L. and D. Von Ludwig: Investment casting for engineers, 79
Boundary layer theory. On the fundamentals of the boundary layer theory (Krzywoblocki), 289(C)
Brain surgery. Brain operations aided by transparent instruments, 474(C)
Brakes. Air pressure warning switch, 512(C)
Bridges. Aluminum bridge for Army, 169(C)
Broadcasting. Multiplex radio transmission, 581(C)
Buckling. On torsional-flexural buckling of columns (Langhaar), 101
Butyl-impregnated leather, 337(N)

C

Calibration. Resonance-type vibration calibrator, 547(N)
Cancer research. Enzymatic system isolated, 544(C)
Carpenter, D. C.: The reaction of formaldehyde with D-threonine, 209
Cast iron. Ductile iron, 332(C)
Casting. Continuous vacuum melting furnaces, 471(C)
Cell chemistry. Enzymatic system isolated, 544(C)
Chang, Chieh-Chien, Boa-Teh Chu and Vivian O'Brien: Asymptotic expansion of the Whittaker's function $W_{k,m}(z)$ for large values of k, m, z, 215, 319
Chu, Boa-Teh, Chieh-Chien Chang and Vivian O'Brien: Asymptotic expansion of the Whittaker's function $W_{k,m}(z)$ for large values of k, m, z, 215, 319
Circuit theory. A contribution to the theory of nonlinear systems (Zadeh), 387
Circuits. Electronic circuits on cloth, 583(C)
Cleaning agents. Refrigerant cleans photographic film, 90(C)
Clutch. A "loudspeaker" clutch, 62(N)
Coaxial cables. New coaxial cable system, 164(C)
Cobalt. New gamma-ray generator, 332(C)
Coffee. Automatic hot coffee server, 92(C)
Color television, 554(C)
Tricolor television tube, 472(C)
Combustion. Rapid gas sampler, 61(N)
Communications. New coaxial cable system, 164(C)
New magnetic material, 544(C)
Computers. Engineering evaluation of flight testing dive or toss bomb computers, 567(F)
High-speed "brains" for high-speed planes, 166(C)
Concrete. Hundred-year tests on strength of concrete, 365(C)
Conductivity. Fundamentals in the behavior of electrets (Swann), 513
Conidendrin-derived compounds tested as stabilizers, 496(C)
Constant-amplitude oscillator, 64(N)
Control devices. New automatic pipetting system, 269(C)
New constant-frequency control, 91(C)
Control test. Development of a production control test, 139(F)
Coordinate systems. Some coordinate systems asociated with elliptic functions (Moon and Spencer), 531
Corrosion. Underground corrosion of galvanized steel, 448(N)
Crystals. Determining orientation of metal crystals with polarized light, 335(N)
Cylindrical coordinate systems. Some coordinate systems associated with elliptic functions (Moon and Spencer), 531

D

D-threonine. The reaction of formaldehyde with D-threonine (Carpenter), 209
Deaf may hear radio and TV, 512(C)
Delay lines. A contribution to the theory of nonlinear systems (Zadeh), 387
Diesel engines. A Diesel engine blower test stand, 8(C)
Distortion. Phonograph-needle drag distortion, 144(N)
Doron. Marine Corps body armor, 267(C)
Dosimeter. Radiation measurement device perfected, 473(C)
Ductile iron, 332(C)
Dust. Smelting furnace dust recovered, 471(C)

E

Elasticity. On torsional-flexural buckling of columns (Langhaar), 101
Elder, Tait and John Strong: The infrared transmission of atmospheric windows, 189
Electrets. Fundamentals in the behavior of electrets (Swann), 513

Electrical networks. The algebraic theory of linear transmission networks (Arsove), 301, 427
Electrodynamics. A "loudspeaker" clutch, 62(N)
Electromechanics. Nonlinear analysis of electro-mechanical problems (Ku), 9
Electronics. An airborne, infrared spectrograph, 59(F)
Constant-amplitude oscillator, 64(N)
Electronic circuits on cloth, 583(C)
Electronic flowmeter, 333(N)
Electronics for defense and industry (Engstrom), 1
Machine for shrinking speeches, 554(C)
New constant-frequency control, 91(C)
Prediction of electronic failures, 140(N)
Robot psychologist, 469(C)
Voice-operated telephone, 269(C)
Electrostatics. Fundamentals in the behavior of electrets (Swann), 513
Elliptic functions. Some coordinate systems associated with elliptic functions (Moon and Spencer), 531
Engineering. Industrial psychiatry increases engineering efficiency, 362(C)
Engines. Rapid gas sampler, 61(N)
Engstrom, E. W.: Electronics for defense and industry, 1
Enzymatic system isolated, 544(C)
Explosions. Slow neutron induced radioactivities of nuclear explosions: III (Mandeville), 133

F

Failure prediction. Prediction of electronic failures, 140(N)
Field theory. Some applications of photic field theory (Moon and Spencer), 113
Theory of the photic field (Moon and Spencer), 33
Film. "Mylar" polyester film, 100(C)
New kind of optical wedges, 268(C)
Refrigerant cleans photographic film, 90(C)
Filters. A contribution to the theory of nonlinear systems (Zadeh), 387
Plastic photographic filters, 496(C)
Transmission-type interference filters, 167(C)
Fire protection. Industry provides own fire protection, 138(C)
Firebrick. Vermiculite insulating firebrick, 32(C)
Flexure. On torsional-flexural buckling of columns (Langhaar), 101
Flowmeter. Electronic flowmeter, 333(N)
Fluid dynamics. Asymptotic expansion of the Whittaker's function $W_{k,m}(z)$ for large values of k, m, z (Chang, Chu and O'Brien), 215, 319
Fluid flow. Electronic flowmeter, 333(N)
On the fundamentals of the boundary layer theory (Krzywoblocki), 289(C)
Fluorography. High voltage fluorography, 352(C)
Fluoroscopy. X-ray image amplifier for medical use, 264(C)
Footwear. Butyl-impregnated leather, 337(N)
Effects of tannage on the properties of leather, 143(N)
Forest fires. Industry provides own fire protection, 138(C)
Formaldehyde. The reaction of formaldehyde with D-threonine (Carpenter), 209
The Franklin Institute
Bartol Research Foundation Notes:
Mandeville, C. E. and H. O. Albrecht: The storage of energy in silver activated potassium chloride, 570
Swann, C. P.: Report on the present status of the high energy Van de Graaff generator constructed at the Bartol Research Foundation of The Franklin Institute, 571
Swann, W. F. G.: The case of charged density distribution *versus* semipermanent polarization as a basis for electret behavior, 569
Swann, W. F. G.: Fundamentals in the behavior of electrets, 570
Weinrich, O. A. and W. E. Danforth: Electrical conduction and thermoelectric power in thorium oxide crystals, 569
Biochemical Research Foundation Notes:
Day, Eugene D. and William A. Mosher: The diazo-coupling reaction for thymine, 455
Ely, J. O., F. A. Tull and J. M. Schanen: The effect of glucosamine on culture of embryo chicken heart cells, 561
Grinnan, Edward L. and William A. Mosher: Cytoplasmic localization of highly polymerized liver ribonucleic acid, 347

Makino, Sajiro: Seminar on cytological phenomena in ascites-sarcomas of rats, 257
McDonald, Ellice: Cancer cell division, foreword, 555
Ross, M. H. and Justine C. Archer: Lack of effect of 2-desoxy-D-glucose on rat liver adenosinetriphosphatase and pyrophosphatase, 161
Ross, M. H., J. O. Ely and Justine Archer: Variation in extractability of phosphatase and nitrogen in liver of rats on protein-varied diets, 87
Woodward, Gladys E. and Marie T. Hudson: D-glucosamine as an antagonist of glucose in carbohydrate metabolism of yeast, 556
By-laws, 170
Committee on Science and the Arts: *Abstract of* Proceedings, December 10, 1952, 71; April 8, 1953, 451; May 13, 1953, 550
Journal of The Franklin Institute, 75, 149, 245, 340, 454, 551
Library, 73, 150, 246, 341, 452, 551
Membership, 68, 148, 248, 343, 451, 550
Museum, 72, 149, 246, 340, 563
Stated Meetings: Minutes, December 17, 1952, 67; January 21, 1953, 147; February 18, 1953, 245; March 18, 1953, 339; April 15, 1953, 450; May 20, 1953, 549
Freon. Refrigerant cleans photographic film, 90(C)
Friction. Operating modes of a servomechanism with non-linear friction (Lauer), 497
Furnaces. Continuous vacuum melting furnaces, 471(C)
Furniture. Modular office furniture, 100(C)

G

Gamma rays. New gamma-ray generator, 332(C)
Gas. New gauge developed for wind tunnel, 365(C)
On the fundamentals of the boundary layer theory (Krzywoblocki), 289(C)
Rapid gas sampler, 61(N)
Viscosimetry of gases at high temperatures and pressures, 445(N)
Gas turbine locomotive, 163(C)
Gas turbines. Viscosimetry of gases at high temperatures and pressures, 445(N)
Gaskets. Rubber gaskets for steel storage tanks, 554(C)
Generators. Genesis of a generator—the early history of the magnetron (Wathen), 271
Liquid-cooled generator, 364(C)
New gamma-ray generator, 332(C)
Grebe, John J.: Nuclear power and industry, 409
Group theory. The algebraic theory of linear transmission networks (Arsove), 301, 427

H

Half life of polonium, 240(N)
Harvey, Richard F.: Development, principles, and applications of interrupted quench hardening, 93
Heat pumps. All-electric heat pump, 332(C)
Heat treating. Development, principles, and applications of interrupted quench hardening (Harvey), 93
Helicopters. One-man helicopter, 473(C)
High polymers. Transitions in Teflon, 545(N)
High-slip motors. New type high-slip motors, 90(C)
History. Genesis of a generator—the early history of the magnetron (Wathen), 271
So, you want to be an inventor (Tuska), 177
Hydro-ski landing gears, 168(C)
Hypersonic flow. On the fundamentals of the boundary layer theory (Krzywoblocki), 289(C)

I

Illumination. Some applications of photic field theory (Moon and Spencer), 113
Theory of the photic field (Moon and Spencer), 33
Induction motors. New type high-slip motor, 90(C)
Industry. Electronics for defense and industry (Engstrom), 1
Infrared transmission. The infrared transmission of atmospheric windows (Elder and Strong), 189
Inspection. High voltage fluorography, 352(C)

Instruments. AEC survey shows growth of radiation instrument industry, 363(C)
An airborne, infrared spectrograph, 59(F)
Alphatron vacuum gauge, 364(C)
Electronic flowmeter, 333(N)
Four watt resistor, 166(C)
Multi-probe Ultra-Viscoson systems, 138(C)
Near-focusing device for close-up photography, 8(C)
New automatic pipetting system, 269(C)
New constant-frequency control, 91(C)
New gauge developed for wind tunnel, 365(C)
New line of vacuum connectors, 165(C)
Prediction of electronic failures, 140(N)
Radiation measurement device perfected, 473(C)
Tachometer generator aids in accurate oil well surveying, 512(C)
Insulation. New type of insulation, 267(C)
Vermiculite insulating firebrick, 32(C)
Inventions. So, you want to be an inventor (Tuska), 177
Inventors. So, you want to be an inventor (Tuska), 177
Iron. Ductile iron, 332(C)
Isothermal heat treating. Development, principles, and applications of interrupted quench hardening (Harvey), 93

J

Jet engines. Viscosimetry of gases at high temperatures and pressures, 445(N)
Jet planes. High-speed "brains" for high-speed planes, 166(C)

K

Kinescope. Tricolor television tube, 472(C)
Kinorama. The kinorama—a device for evaluating airport approach lights, 237(N)
"Kordaroom." Modular office furniture, 100(C)
Krzywoblocki, M. Z.: On the fundamentals of the boundary layer theory, 289
Ku, Y. H.: Nonlinear analysis of electromechanical problems, 9

L

Langhaar, H. L.: On torsional-flexural buckling of columns, 101
Lauer, Henri: Operating modes of a servomechanism with non-linear friction, 497
Leather. Butyl-impregnated leather, 337(N)
Effects of tannage on the properties of leather, 143(N)
Lenses. Wide-angle aerial lens, 169(C)
Light. New kind of optical wedges, 268(C)
Some applications of photic field theory (Moon and Spencer), 113
Theory of the photic field (Moon and Spencer), 33
Lightning. Artificial lightning tests, 469(C)
Liquid-cooled generator, 364(C)
Lithography. Air dampening system for lithography, 472(C)
Locomotive. Gas turbine locomotive, 163(C)
"Loudspeaker" clutch, 62(N)

M

Magnetic amplifier. High-speed "brains" for high-speed planes, 166(C)
Magnetic materials. New magnetic material, 544(C)
Magnetron. Genesis of a generator—the early history of the magnetron (Wathen), 271
Management. Automatic hot coffee server, 92(C)
Industrial psychiatry increases engineering efficiency, 362(C)
Modular office furniture, 100(C)
Mandeville, C. E.: Slow neutron induced radioactivities of nuclear explosions: III, 133
Map making. Air dampening system for lithography, 472(C)
New plastic scribing process, 266(C)
Martempering. Development, principles, and applications of interrupted quench hardening (Harvey), 93
Masonry. Hundred-year tests on strength of concrete, 365(C)
Mass testing. Robot psychologist, 469(C)
Mathematical analysis. Asymptotic expansion of the Whittaker's function $W_{k,m}(z)$ for large values of k, m, z (Chang, Chu and O'Brien), 215, 319
Mathematics. On the fundamentals of the boundary layer theory (Krzywoblocki), 289(C)
Measurement. Alphatron vacuum gauge, 364((C)

Multi-probe Ultra-Viscoson systems, 138(C)
Medicine. Brain operations aided by transparent instruments, 474(C)
What causes the "itch" in poison ivy?, 266(C)
X-ray image amplifier for medical use, 264(C)
Metallurgy. Continuous vacuum melting furnaces, 471(C)
Determining orientation of metal crystals with polarized light, 335(N)
Ductile iron, 332(C)
Meteorology. The infrared transmission of atmospheric windows (Elder and Strong), 189
Microwave vacuum tubes. Genesis of a generator—the early history of the magnetron (Wathen), 271
Military engineering. All-wood truck body, 91(C)
Aluminum bridge for Army, 169(C)
Army reclaims old ammunition, 8(C)
Electronics for defense and industry (Engstrom), 1
The engineering development of the 280-mm. weapon, 565(F)
Lightweight triangulation tower, 288(C)
New plastic scribing process, 266(C)
Marine Corps body armor, 267(C)
Prediction of electronic failures, 140(N)
Mining. Unique mine-ventilation system, 362(C)
Moon, Parry and Domina Eberle Spencer: Some applications of photic field theory, 113
Some coordinate systems associated with elliptic functions, 531
Theory of the photic field, 33
Motors. New type high-slip motor, 90(C)
"Mylar" polyester film, 100(C)

N

Nature. Is the universe planned? (Swann), 367
Network analyzer. An a-c network analyzer, 344(F)
Network theory. The algebraic theory of linear transmission networks (Arsove), 301, 427
Neutrons. Slow neutron induced radioactivities of nuclear explosions: III (Mandeville), 133
Nonlinear analysis. Nonlinear analysis of electro-mechanical problems (Ku), 9
Nonlinear systems. A contribution to the theory of nonlinear systems (Zadeh), 387
Nuclear physics. "Hot lab"—atomic age tool, 167(C)
Nuclear power and industry (Grebe), 409
Nuclear radiation. Slow neutron induced radioactivities of nuclear explosions: III (Mandeville), 133
Number theory. Radio technology and the theory of numbers (van der Pol), 475

O

O'Brien, Vivian, Chieh-Chien Chang and Boa-Teh Chu: Asymptotic expansion of the Whittaker's function $W_{k,m}(z)$ for large values of k, m, z, 215, 319
Oceanography. Aqualung, 288(C)
Oil. Tachometer generator aids in accurate oil well surveying, 512(C)
Operating modes of a servomechanism with non-linear friction (Lauer), 497
Optics. New kind of optical wedges, 268(C)
Plastic photographic filters, 496(C)
Some applications of photic field theory (Moon and Spencer), 113
"Supersonic" eyes, 352(C)
Theory of the photic field (Moon and Spencer), 33
Transmission-type interference filters, 167(C)
Ordnance. Army reclaims old ammunition, 8(C)
Orthogonal mapping. Some coordinate systems associated with elliptic functions (Moon and Spencer), 531
Oscillators. Constant-amplitude oscillator, 64(N)

P

Patents. So, you want to be an inventor (Tuska), 177
Philosophy. Is the universe plannned? (Swann), 367
Phonograph-needle drag distortion, 144(N)
Photic field. Some applications of photic field theory (Moon and Spencer), 113
Theory of the photic field (Moon and Spencer), 33
Photoelasticity. Polarized light: a research tool, 242(F)

Photography. Elementary probability of efficient absorption of a quantum proportional to the volume of a grain (Silberstein), 51
Near-focusing device for close-up photography, 8(C)
Photographic typesetting developed, 473(C)
Plastic photographic filters, 496(C)
Refrigerant cleans photographic film, 90(C)
Wide-angle aerial lens, 169(C)
Photometry. Some applications of photic field theory (Moon and Spencer), 113
Theory of the photic field (Moon and Spencer), 33
Transmission-type interference filters, 167(C)
Pipe. Underground corrosion of galvanized steel, 448(N)
Pipetting. New automatic pipetting system, 269(C)
Plastics. Brain operations aided by transparent instruments, 474(C)
New plastic scribing process, 266(C)
Plastic photographic filters, 496(C)
Transitions in Teflon, 545(N)
Poison ivy. What causes the "itch" in poison ivy?, 266(C)
Polarization. Fundamentals in the behavior of electrets (Swann), 513
Polarized light. Determining orientation of metal crystals with polarized light, 335(N)
Polarized light: a research tool, 242(F)
Polonium. Half life of polonium, 240(N)
Polyester film. "Mylar" polyester film, 100(C)
Polyisobutylene. Butyl-impregnated leather, 337(N)
Power systems. An a-c network analyzer, 344(F)
Nuclear power and industry (Grebe), 409
Printing. Air dampening system for lithography, 472(C)
Photographic typesetting developed, 473(C)
Probability theory. Elementary probability of efficient absorption of a quantum proportional to the volume of a grain (Silberstein), 51
Psychiatry. Industrial psychiatry increases engineering efficiency, 362(C)
Psychology. Robot psychologist, 469(C)
Pumps. All-electric heat pump, 332(C)

Q

Quantum theory. Elementary probability of efficient absorption of a quantum proportional to the volume of a grain (Silberstein), 51

R

Radiation. AEC survey shows growth of radiation instrument industry, 363(C)
Radio. Deaf may hear radio and TV, 512(C)
Multiplex radio transmission, 581(C)
Radio technology and the theory of numbers (van der Pol), 475
Radioactivity. Half life of polonium, 240(N)
"Hot lab"—atomic age tool, 167(C)
Improved self-luminous markers developed, 474(C)
New gamma-ray generator, 332(C)
Radiation measurement device perfected, 473(C)
Radioactive waste gases, 165(C)
Slow neutron induced radioactivities of nuclear explosions: III (Mandeville), 133
Radiology. X-ray image amplifier for medical use, 264(C)
Railroads. Gas turbine locomotive, 163(C)
Rayon. Heated knives cut rayon strip without fraying, 32(C)
Reactors. Nuclear power and industry (Grebe), 409
Recording. Machine for shrinking speeches, 554(C)
Refrigerant cleans photographic film, 90(C)
Reproduction. Near-focusing device for close-up photography, 8(C)
Resistors. Four watt resistor, 166(C)
Rotational coordinate systems. Some coordinate systems associated with elliptic functions (Moon and Spencer), 531
Rubber. Rubber gaskets for steel storage tanks, 554(C)

S

Safes. Giant safe—will protect precious documents, 168(C)
Safety engineering. Air pressure warning switch, 512(C)
Giant safe—will protect precious documents, 168(C)
"Hot lab"—atomic age tool, 167(C)

The kinorama—a device for evaluating airport approach lights, 237(N)
Radioactive waste gases, 165(C)
Sampling. Rapid gas sampler, 61(N)
Sawtooth function. Radio technology and the theory of numbers (van der Pol), 475
Servomechanisms. Operating modes of a servo-mechanism with non-linear friction (Lauer), 497
Ship repair. A mobile woodworking shop, 168(C)
Silberstein, Ludwik: Elementary probability of efficient absorption of a quantum proportional to the volume of a grain, 51
Simulated service testing. Artificial lightning tests, 469(C)
A Diesel engine blower test stand, 8(C)
Hundred-year tests on strength of concrete, 365(C)
New axle tester, 300(C)
Smelting furnace dust recovered, 471(C)
Sound. Phonograph-needle drag distortion, 144(N)
Spectrograph. An airborne, infrared spectrograph, 59(F)
Spencer, Domina Eberle and Parry Moon: Some applications of photic field theory, 113
Some coordinate systems associated with elliptic functions, 531
Theory of the photic field, 33
Stabilizers. Conidendrin-derived compounds tested as stabilizers, 496(C)
Steel. Development, principles and applications of interrupted quench hardening (Harvey), 93
Underground corrosion of galvanized steel, 448(N)
Step quenching. Development, principles, and applications of interrupted quench hardening (Harvey), 93
Storage batteries. Development of a production control test, 139(F)
Storage tanks. Rubber gaskets for steel storage tanks, 554(C)
Stress analysis. Polarized light: a research tool, 242(F)
Strong, John and Tait Elder: The infrared transmission of atmospheric windows, 189
Strontium. Improved self-luminous markers developed, 474(C)
Surge generators. Artificial lightning tests, 469(C)
Surveying. Tachometer generator aids in oil well surveying, 512(C)
Swann, W. F. G.: Fundamentals in the behavior of electrets, 513
Is the universe planned?, 367
Synchronous machines. Nonlinear analysis of electro-mechanical problems (Ku), 9

T

Tanning. Effects of tannage on the properties of leather, 143(N)
Teflon. Transitions in Teflon, 545(N)
Telephony. New coaxial cable system, 164(C)
Voice-operated telephone, 269(C)
Television. Color television, 554(C)
Deaf may hear radio and TV, 512(C)
Tricolor television tube, 472(C)
Testing. Artificial lightning tests, 469(C)
Engineering evaluation of flight testing dive or toss bomb computers, 567(F)
Hundred-year tests on strength of concrete, 365(C)
New axle tester, 300(C)
Textiles. Electronic circuits on cloth, 583(C)
Heated knives cut rayon strip without fraying, 32(C)
Theory of columns. On torsional-flexural buckling of columns (Langhaar), 101
Theory of numbers. Radio technology and the theory of numbers (van der Pol), 475
Threonine. The reaction of formaldehyde with D-threonine (Carpenter), 209
Torsion. On torsional-flexural buckling of columns (Langhaar), 101
Transformers. 400-ampere a-c. welding transformer, 474(C)
Transmission. Machine for shrinking speeches, 554(C)
Multiplex radio transmission, 581(C)
Transmission networks. The algebraic theory of linear transmission networks (Arsove), 301, 427
Triangulation tower. Lightweight triangulation tower, 288(C)
Trucks. All-wood truck body, 91(C)
New axle tester, 300 (C)
Turbines. Liquid-cooled generator, 364(C)

Tuska, C. D.: So, you want to be an inventor, 177
Two-poles. A contribution to the theory of nonlinear systems (Zadeh), 387
Typesetting. Photographic typesetting developed, 473(C)

U

Universe. Is the universe planned? (Swann), 367

V

Vacuum gauge. Alphatron vacuum gauge, 364(C)
Vacuum systems. New line of vacuum connectors, 165(C)
Vacuum tubes. Genesis of a generator—the early history of the magnetron (Wathen), 271
van der Pol, Balth.: Radio technology and the theory of numbers, 475
Ventilation. Unique mine-ventilation system, 362(C)
Vermiculite insulating firebrick, 32(C)
Vibration. Resonance-type vibration calibrator, 547(N)
Viscosity. Multi-probe Ultra-Viscoson systems, 138(C)
Viscosimetry of gases at high temperatures and pressures, 445(N)
Vision. "Supersonic" eyes, 352(C)
Voice-operated telephone, 269(C)

W

Waste. Radioactive waste gases, 165(C)
Smelting furnace dust recovered, 471(C)
Wathen, Robert L.: Genesis of a generator—the early history of the magnetron, 271
Welding. An air-cooled welding torch, 168(C)
400-ampere a-c. welding transformer, 474(C)
Whittaker's function. Asymptotic expansion of the Whittaker's function $W_{k,m}(z)$ for large values of k, m, z (Chang, Chu and O'Brien), 215, 319
Wind tunnel. New gauge developed for wind tunnel, 365(C)
Wood. All-wood truck body, 91(C)
A mobile woodworking shop, 168(C)

X

X-ray. X-ray image amplifier for medical use, 264(C)

Z

Zadeh, L. A.: A contribution to the theory of nonlinear systems, 387
Zinc. Underground corrosion of galvanized steel, 448(N)